Notes for Section 2.4, #21

We didn't get a chance to finish the solution to exercise 21 in class, so I thought I should write up a more formal solution.

The problem was to consider $y' = y^{1/3}$, with $y(0) = 0$.

As we discussed in class, we can come up with a couple of solutions without too much work- $y(t) = 0$ and $y(t) = (\frac{2}{3}t)^{3/2}$ were the ones we had. The exercise also mentions "Example 3", from which we have even more solutions, for any $t_0 > 0$:

$$y(t) = \begin{cases} 0 & \text{if } t < t_0 \\ \pm(\frac{2}{3}(t - t_0))^{3/2} & \text{if } t \geq 0 \end{cases}$$

The first part of the problem asks us to find a solution that also goes through $(1, 1)$. If we use the general form, that would mean:

$$1 = \left(\frac{2}{3}(1 - t_0)\right)^{3/2} \quad \Rightarrow \quad \frac{3}{2} = 1 - t_0 \quad \Rightarrow \quad t_0 = -\frac{1}{2}$$

Because $t_0 > 0$, this would not yield a valid solution. Our other solutions also do not go through $(1, 1)$, so therefore, there is no solution at all for this.

Now, how about a solution going through $(2, 1)$? In that case,

$$1 = \left(\frac{2}{3}(2 - t_0)\right)^{3/2} \quad \Rightarrow \quad \frac{3}{2} = 2 - t_0 \quad \Rightarrow \quad t_0 = \frac{1}{2}$$

Therefore, the solution is from the general form with $t_0 = 1/2$.

In the final part, consider all possible solutions to the IVP. If we want a solution to go through $(y_0, 2)$, is y_0 restricted, or can it be any number? Doing a similar computation to the one before,

$$y_0 = \left(\frac{2}{3}(2 - t_0)\right)^{3/2} \quad \Rightarrow \quad \frac{3}{2}y_0^{2/3} = 2 - t_0 \quad \Rightarrow \quad t_0 = 2 - \frac{3}{2}y_0^{2/3}$$

We would require $t_0 > 0$, so that does put a restriction on y_0:

$$2 - \frac{3}{2}y_0^{2/3} > 0 \quad \Rightarrow \quad \frac{3}{2}y_0^{2/3} < 2 \quad \Rightarrow \quad y_0^{2/3} < \frac{4}{3}$$

Note that y_0 could be negative, so raising both sides to the 3/2 power, take the absolute value:

$$|y_0| < (4/3)^{3/2}$$

Lots of algebra for that one!