## 3.7 Summary

Some trig to start things off:

If we want to write $A\cos(\omega t) + B\sin(\omega t)$ as $R\cos(\omega t - \delta)$, we take

$$R = \sqrt{A^2 + B^2} \qquad \delta = \tan^{-1}(B/A)$$

As a memory device, you might note that these quantities represent represent the polar form of $A + Bi$.

# The General Non-Forced Model

Solve $mu'' + \gamma u' + ku = F(t)$

### No forcing, No damping

With no damping and no forcing, the model is then: $mu'' + ku = 0$. The general solution is:

$$u(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) = R\cos\left(\sqrt{\frac{k}{m}}\, t - \delta\right)$$

*CONCLUSION:* With no damping and no forcing, the spring-mass system simply oscillates. Its amplitude and period is given by

$$R = \sqrt{C_1^2 + C_2^2} \qquad \text{Period} = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}} \qquad \delta = \arctan\left(\frac{C_2}{C_1}\right)$$

Or we could say that the circular frequency is $\sqrt{k/m}$.

### Adding in Damping

By adding in the damping constant $\gamma$, we have the following roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Consider the three cases:

- $\gamma^2 - 4mk > 0$. This is called "**OVERDAMPED**", as all solutions go to zero pretty quickly (we have two exponentials).
- $\gamma^2 - 4mk = 0$. Then $r = -\gamma/2m$. The solution is then "**CRITICALLY DAMPED**". They go to zero, but perhaps not as quickly.
- $\gamma^2 - 4mk < 0$. Then we have complex roots (with sine/cosine). The solution oscillates as it goes to zero in amplitude (UNDERDAMPED). Not periodic, but "quasiperiodic".

We would note that in all cases, if there is damping, all solutions tend to zero. Some solutions go pretty quickly, with no oscillations, and some will still oscillate a bit, with smaller and smaller amplitude.