Homework for 3.7

1. For practice with trig, 3.7.1-3.7.4, also given below. In each case, write the sum as $R\cos(\omega t - \delta)$

 3.7.1 $3\cos(2t) + 4\sin(2t)$
 SOLUTION: For each of these, think of $A\cos(2t)+B\sin(2t)$ as defining a complex number $A + iB$. Then R is the magnitude and δ is the angle for $A + iB$. In this particular case, we see that (A, B) is in Quadrant I, so δ does not need an extra π added to it:

$$R = \sqrt{9+16} = 5 \qquad \delta = \tan^{-1}(4/3) \Rightarrow$$
$$3\cos(2t) + 4\sin(2t) = 5\cos(2t - \tan^{-1}(4/3))$$

 3.7.2 $-\cos(t) + \sqrt{3}\sin(t)$
 SOLUTION: Note that in this case, $(-1, \sqrt{3})$ is in Quadrant II, so add π to δ. Also, notice that the angle δ is coming from a triangle with side $1, 2, \sqrt{3}$ (or 30-60-90). In this case,

$$R = \sqrt{1+3} = \sqrt{2} \qquad \delta = \tan^{-1}(-\sqrt{3}) = -\pi/3$$
$$-\cos(t) + \sqrt{3}\sin(t) = 2\cos(t - (2\pi/3))$$

 3.7.3 $4\cos(3t) - 2\sin(3t)$
 SOLUTION: In this case, $(4, -2)$ is coming from Quadrant IV, so no need to add π to δ. We don't have a special triangle in this case.

$$R = \sqrt{16+4} = 2\sqrt{5} \qquad \delta = \tan^{-1}(-1/2)$$
$$4\cos(3t) - 2\sin(3t) = 2\sqrt{5}\cos(t - \tan^{-1}(-1/2))$$

 3.7.4 $-2\cos(\pi t) - 3\sin(\pi t)$
 SOLUTION: In this case $(-2, -3)$ is in Quadrant III, so we'll need to add π to δ.

$$R = \sqrt{4+9} = \sqrt{13} \qquad \delta = \tan^{-1}\left(\frac{3}{2}\right) + \pi$$

2. Practice with the Model (metric system):

 (a) A spring with a 3-kg mass is held stretched 0.6 meters beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds (assume no damping).

SOLUTION: $m = 3$, $\gamma = 0$, so we just need the spring constant k. By Hooke's Law, the force is proportional to the length stretched:

$$k(0.6) = 20 \quad \Rightarrow \quad k = \frac{100}{3}$$

Now we have the IVP:

$$3u'' + \frac{100}{3}u = 0 \qquad u(0) = 0, \quad u'(0) = \frac{6}{5}$$

To solve this,

$$3r^2 + \frac{100}{3} = 0 \qquad r = \sqrt{\frac{100}{9}}\,i = \frac{10}{3}\,i$$

The general solution is

$$C_1 \cos\left(\frac{10}{3}\,t\right) + C_2 \sin\left(\frac{10}{3}\,t\right)$$

Putting in the initial conditions,

(b) A spring with a 4-kg mass has a natural length of 1 meter, and is maintained stretched to a length of 1.3 m by a force of 24.3 N. If the spring is compressed to a length of 0.8 m and then released with zero velocity, find the position of the mass after t seconds (assume no damping).

SOLUTION: $m = 4$, $\gamma = 0$, and the wording of the question implies that the spring is stretched 0.3 m beyond natural length, giving (convert to fractions)

$$\frac{3k}{10} = \frac{243}{10} \quad \Rightarrow \quad k = 81$$

Therefore, we have (recall that "up" is negative)

$$4u'' + 81u = 0 \qquad u(0) = -\frac{4}{5} \quad u'(0) = 0$$

Now solve $u'' + \frac{81}{4}u = 0$:

$$u(t) = C_1 \cos\left(\frac{9}{2}\,t\right) + C_2 \sin\left(\frac{9}{2}\,t\right))$$

Solving for the constants, you should get $C_1 = -4/5, C_2 = 0$.

(c) A spring with a mass of 2 kg has damping constant 14 and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is then stretched to 1 m beyond its natural length and released. Find the position of the mass at any time t.

SOLUTION: $m = 2$, $\gamma = 14$ and for the spring constant, we have:

$$\frac{k}{2} = 6 \quad \Rightarrow \quad k = 12$$

The IVP is: $2u'' + 14u' + 12u = 0$, with $u(0) = 1$ and $u'(0) = 0$

$$u(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$$

(d) A spring with a mass of 3 kg has a damping constant 30 and spring constant 123. Find the position of mass at time t if it starts at equilibrium with a velocity of 2 m/s.
SOLUTION:

$$3u'' + 30u' + 123u = 0 \qquad u(0) = 0 \quad u'(0) = 2$$

Solving, (divide everything by 3 first) we get $u(t) = \frac{1}{2}e^{-5t}\sin(4t)$.

(e) For the spring model above with a mass of 4 kg, find the damping constant that would produce critical damping.
SOLUTION:

$$4u'' + \gamma u' + 123u = 0$$

For critical damping, $\gamma^2 - 4(4)(123) = 0$, so that $\gamma = 4\sqrt{123}$ (only take the positive root!)

(f) A mass of 20 grams stretches a spring 5 cm. Suppose tha the mass is attached to a viscous damper with a constant damping constant of 400 dyn-s/cm (note: a dyne is a unit of force using centimeters-grams- seconds for units). If the mass is pulled down an additional 2 cm then released, find the IVP that governs the motion of the mass. (Calculator needed- g should be taken as 980).
SOLUTION: In this case, the spring is stretched to equilibrium:

$$mg - kL = 0 \quad \Rightarrow \quad (20)(980) - k(5) = 0 \quad \Rightarrow \quad k = 3920$$

Therefore, the DE is:

$$20u'' + 400u' + 3920u = 0 \qquad u(0) = 2, \quad u'(0) = 0$$

(Note that we could divide everything by 20 to make it a bit easier).

(g) Suppose we consider a mass-spring system with no damping (the damping constant is then 0), so that the differential equation expressing the motion of the mass can be modeled as

$$mu'' + ku = 0$$

Find value(s) of β so that $A\cos(\beta t)$ and $B\sin(\beta t)$ are each solutions to the homogeneous equation (for arbitrary values of A, B).
SOLUTION: $\beta = \sqrt{\frac{k}{m}}$

3. Practice with the model (in pounds, from the text)

(a) A mass weighing 4 lb stretches a spring 2 in. Suppose that the mass is displaced an additional 6 in the positive direction and released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Formulate the IVP that governs the motion of the mass. (Hint: All units should be consistent. When working with US units, use pounds, feet and seconds.)

SOLUTION: This is Example 1, Section 3.7. For mass, $mg = 4$, so

$$m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8}$$

We assume that the damping force is proportional to the velocity, so what is given translates to:

$$6 = \gamma u' = \gamma(3) \quad \Rightarrow \quad \gamma = 2$$

And the spring constant: Change inches into feet so that units are consistent, and

$$mg - kL = 0 \quad \Rightarrow \quad 4 - k\frac{2}{12} = 0 \quad \Rightarrow \quad k = 24$$

Now we have

$$\frac{1}{8}u'' + 2u' + 24u = 0 \quad u(0) = \frac{1}{2} \qquad u'(0) = 0$$

(b) A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the IVP that governs the motion of the mass. (You might re-read the hint for (1))

SOLUTION: (Exercise 5, 3.7)

From what's given, $mg = 2$ and $mg - kL = 2 - k/2 = 0$, so $m = 2/32 = 1/16$ and $k = 4$:

$$\frac{1}{16}u'' + 4u = 0 \qquad u(0) = \frac{1}{4} \qquad u'(0) = 0$$

(c) (Repeat of 1(f))

(d) A mass weighing 8 lbs stretches a spring $\frac{3}{2}$ in. The mass is attached to a damper with coefficient γ. Find γ so that the spring is *underdamped*, *critically damped*, *overdamped*.

SOLUTION: $mg = 8$, so $m = 8/32 = 1/4$. Further,

$$8 - \frac{3k}{24} = 0 \quad \Rightarrow \quad k = 64$$

The model is then:

$$\frac{1}{4}u'' + \gamma u' + 64u = 0$$

The discriminant in the quadratic formula is $\gamma^2 - 4mk = \gamma^2 - 64$:

- If $\gamma > 8$, then OVERDAMPED.
- If $\gamma = 8$, then CRITICALLY DAMPED.
- If $0 \leq \gamma < 8$, then UNDERDAMPED.