Quiz 8 (Take Home) Solutions

1. Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$

   SOLUTION: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{n3^n}{(n+1)3^{n+1}}|x-1| = \lim_{n\to\infty} \frac{n}{n+1}\frac{1}{3}|x-1| = \frac{|x-1|}{3} < 1 \quad \Rightarrow \quad |x-1| < 3$$

   The radius is 3. Check the endpoints, $x = 4$ and $x = -2$.

   At $x = 4$, the sum becomes the (divergent) harmonic series:

$$\sum_{n=1}^{\infty} \frac{(4-1)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

   At $x = -2$, the sum becomes the (convergent) alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-2-1)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

   Therefore, the interval of convergence is $[-2, 4)$.

2. Exercise #24 in 5.1. We want to rewrite the given expression as a sum whose generic term involves x^n.

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

   Now, to bring the sums together, they need to start with the same power. The first sum starts with x^0, the second sum starts with x^2. However, in the second term, we are able to simply start the sum at $n = 0$ instead, because the first two terms would remain zero:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

   Writing the generic sum as $\sum_{m=0}^{\infty}(?)x^m$, we see that in the first sum, $m = n-2$ and in the second sum, $m = n$. Putting these together,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n = \sum_{m=0}^{\infty} ((m+2)(m+1)a_{m+2} - m(m+1)a_m)x^m$$

3. For the IVP in #8, Section 5.2, find the recurrence relation (as we did in the class notes).

We substitute the following into the given IVP, $xy'' + y' + xy = 0$, with $y(0) = a_0$ and $y'(0) = a_1$.

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

We get:

$$x \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

In order to incorporate x into the sum, we'll need to write $x = (x-1)+1$ in front of y'' and in front of y. Notice that this will create 5 sums- Two from the second derivative, one from the first, and two for the y term.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} +$$

$$\sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Normally we could make all the sum begin with $(x-1)^0$, but the fourth sum will prevent this. Instead, we'll start every sum with $(x-1)^1$, removing the zeroth term if needed. That is, we'll rewrite each sum (the five sums are below):

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} \text{ Ready as is... Let } m = n-1$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n (x-1)^{n-2} \text{ Let } m = n-2$$

$$\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n (x-1)^{n-1} \text{ Let } m = n-1$$

$$\sum_{n=0}^{\infty} a_n (x-1)^{n+1} \text{ Ready as is... Let } m = n+1$$

$$\sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-1)^n \text{ Let } m = n$$

Putting them all together (sorry about breaking it up):

$$a_0 + a_1 + 2a_2 +$$

$$\sum_{m=1}^{\infty} \left((m+1) m a_{m+1} + (m+2)(m+1) a_{m+2} + (m+1) a_{m+1} + a_{m-1} + a_m \right) (x-1)^m = 0$$

A little more algebra- Simplifying the terms with a_{m+1}, we have:

$$(m+1)ma_{m+1} + (m+1)a_{m+1} = (m+1)a_{m+1}(m+1) = (m+1)^2 a_{m+1}$$

Therefore, the general recurrence is given by:

$$a_{m+2} = \frac{-(m+1)^2 a_{m+1} - a_m - a_{m-1}}{(m+2)(m+1)}$$

4. For $y'' + \sin(x)y' + \cos(x)y = 0$, find the first five terms of the power series expansion for $y(x)$ as we did in the first section of these notes. (Note: Zeros do count as a term).

   We have $y(0) = 0$ and $y'(0) = 1$. For the remaining terms:

   SOLUTION:

$$y'' = -\sin(x)y' - \cos(x)y \quad \Rightarrow \quad y''(0) = 0 - 0 = 0$$

$$y''' = -\cos(x)y' - \sin(x)y'' + \sin(x)y - \cos(x)y' = -2\cos(x)y' + \sin(x)(y - y'') \quad \Rightarrow \quad y'''(0) = -2$$

   Similarly,

$$y^{(4)} = 2\sin(x)y' - 2\cos(x)y'' + \cos(x)(y - y'') + \sin(x)(y' - y''') = 0$$

   Therefore, we have:

$$y(x) = 0 + x + 0 - \frac{2}{3!}x^3 + 0 + \cdots$$