Exam 2 Summary of Topics

The exam will cover material from Section 3.1 to 3.8. You will be allowed to use your notes, the textbook, a calculator, and anything on the class website to help you.

Structure and Theory (Mostly 3.2)

The goal of the theory was to establish the structure of solutions to the second order IVP:

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0$$

We saw that two functions form a fundamental set of solutions to the homogeneous DE if the Wronskian is not zero at t_0.

1. Vocabulary: Linear operator, general solution, fundamental set of solutions, linear combination of a set of functions.
2. Theorems:
   - The Existence and Uniqueness Theorem for $y'' + p(t)y' + q(t)y = g(t)$.
   - Principle of Superposition.
   - Abel's Theorem.
     If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian, $W(y_1, y_2)$, is either always zero or never zero on the interval for which the solutions are valid.
     That is because the Wronskian may be computed as:
     $$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\,dt}$$
   - The Fundamental Set of Solutions: $y'' + p(t)y' + q(t)y = 0$
     We can guarantee that we can always find a fundamental set of solutions (where p, q are continuous). We did that by appealing to the Existence and Uniqueness Theorem for the following two initial value problems:
     - y_1 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 1, y'(t_0) = 0$
     - y_2 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 0, y'(t_0) = 1$
3. The Structure of Solutions to $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0, y'(t_0) = v_0$
   Given a fundamental set of solutions to the homogeneous equation, y_1, y_2, then there is a solution to the initial value problem, written as:
   $$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$$
   where $y_p(t)$ solves the non-homogeneous equation.

   In fact, if we have:
   $$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t),$$
   we can solve by splitting the problem up into smaller problems:
   - y_1, y_2 form a fundamental set of solutions to the homogeneous equation.
   - y_{p_1} solves $y'' + p(t)y' + q(t)y = g_1(t)$
   - y_{p_2} solves $y'' + p(t)y' + q(t)y = g_2(t)$
     and so on..
   - y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$

   and the full solution is:
   $$y(t) = C_1y_1 + C_2y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$$

Finding the Homogeneous Solution

We had two distinct equations to solve-

$$ay'' + by' + cy = 0 \qquad \text{or} \qquad y'' + p(t)y' + q(t)y = 0$$

First we look at the case with constant coefficients, then we look at the more general case.

Constant Coefficients

To solve

$$ay'' + by' + cy = 0$$

we use the **ansatz** $y = \mathrm{e}^{rt}$. Then we form the associated **characteristic equation**:

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way:

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2. The general solution is:

  $$y_h(t) = c_1 \mathrm{e}^{r_1 t} + c_2 \mathrm{e}^{r_2 t}$$

  If $a, b, c > 0$ (as in the Spring-Mass model) we can further say that r_1, r_2 are negative. We would say that this system is OVERDAMPED.

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

  $$y_h(t) = \mathrm{e}^{-(b/2a)t}\,(C_1 + C_2 t)$$

  If $a, b, c > 0$ (as in the Spring-Mass model), the exponential term has a negative exponent. In this case (one real root), the system is CRITICALLY DAMPED.

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \alpha \pm i\beta$. Then the solution is:

  $$y_h(t) = \mathrm{e}^{\alpha t}\,(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

  If $a, b, c > 0$, then $\alpha = -(b/2a) < 0$. In the case of complex roots, the system is said to the UNDERDAMPED. If $\alpha = 0$ (this occurs when there is no damping), we get pure periodic motion, with period $2\pi/\beta$ or circular frequency β.

Solving the more general case

We had two methods for solving the more general equation:

$$y'' + p(t)y' + q(t)y = 0$$

but each method relied on already having one solution, $y_1(t)$. Given that situation, we can solve for y_2 (so that y_1, y_2 form a fundamental set), by one of two methods:

- By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$ (This is from Abel's Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = C\mathrm{e}^{-\int p(t)\,dt}$

- Reduction of order: Given that y_1 solves the homog DE, we look for a second solution, y_2. We assume $y_2 = v(t) y_1(t)$. Now substitute y_2 into the DE, and use the fact that y_1 solves the homogeneous equation, and the DE reduces to:

  $$y_1 v'' + (2y_1' + p y_1) v' = 0$$

Finding the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters.

Method of Undetermined Coefficients

This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, the table from the text:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1 t + \dots a_n t^n)$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t}(a_0 + a_1 t + \dots + a_n t^n)$
$P_n(t)e^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s e^{\alpha t}\left((a_0 + a_1 t + \dots + a_n t^n)\sin(\mu t) + (b_0 + b_1 t + \dots + b_n t^n)\cos(\mu t)\right)$

The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

Variation of Parameters:

Given $y'' + p(t)y' + q(t)y = g(t)$, with y_1, y_2 solutions to the homogeneous equation, we write the ansatz for the particular solution as:

$$y_p = u_1 y_1 + u_2 y_2$$

From our analysis, we saw that u_1, u_2 were required to solve:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned} \qquad \text{Cramer's Rule} \qquad \Rightarrow \quad u_1' = \frac{-y_2 g}{W(y_1, y_2)} \qquad u_2' = \frac{y_1 g}{W(y_1, y_2)}$$

Analysis of the Oscillator Model

1. Unforced: $mu'' + \gamma u' + ku = 0$
   (a) No damping, $\gamma = 0$: Natural frequency is $\sqrt{k/m}$
   (b) With damping, $\gamma > 0$: Underdamped, Critically Damped, Overdamped
2. Periodic Forcing: $mu'' + \gamma u' + ku = F_0 \cos(\omega t)$
   (a) No damping: When does beating, resonance occur: $u'' + \omega_0^2 u = F\cos(\omega t)$.
   "Beating" occurs when ω is close to ω_0. What is the period of one beat?
   "Resonance" occurs when $\omega = \omega_0$. The solution becomes unbounded.
   (b) With damping: Be able to solve using complexification.
   - Find just the amplitude and phase angle for the particular solution only.
   - Find the ω that maximizes the amplitude of the forced response (or particular part).

Other Material

1. Be familiar with complex numbers, their polar form, and basic operations using complex numbers.
2. Know and use Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
3. Be able to write
$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$