Exercise Set 3 (HW to replace 7.3, 7.5)

This homework is all about solving for eigenvalues and eigenvectors, and we'll also do some visualization and classification of equilibria.

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

SOLUTION: It's probably best to break it apart-

$$\begin{array}{l} x_1(t) = C_1 \mathrm{e}^{-t} + 2C_2 \mathrm{e}^{2t} \\ x_2(t) = 2C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{2t} \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1'(t) = -C_1 \mathrm{e}^{-t} + 4C_2 \mathrm{e}^{2t} \\ x_2'(t) = -2C_1 \mathrm{e}^{-t} + 2C_2 \mathrm{e}^{2t} \end{array}$$

Verify that these expressions for x_1', x_2' are indeed found by

$$x_1' = 3x_1 - 2x_2 \quad \text{and} \quad x_2' = 2x_1 - 2x_2$$

For example, the first expression below should be x_1' and the second should be x_2'

$$\begin{array}{rl} 3x_1(t) & = 3C_1 \mathrm{e}^{-t} + 6C_2 \mathrm{e}^{2t} \\ -2x_2(t) & = -4C_1 \mathrm{e}^{-t} - 2C_2 \mathrm{e}^{2t} \\ \hline & = -C_1 \mathrm{e}^{-t} + 4C_2 \mathrm{e}^{2t} \end{array} \qquad \begin{array}{rl} 2x_1(t) & = 2C_1 \mathrm{e}^{-t} + 4C_2 \mathrm{e}^{2t} \\ -2x_2(t) & = -4C_1 \mathrm{e}^{-t} - 2C_2 \mathrm{e}^{2t} \\ \hline & = -2C_1 \mathrm{e}^{-t} + 2C_2 \mathrm{e}^{2t} \end{array}$$

And those do check out.

2. For each matrix, find the eigenvalues and eigenvectors, then solve the corresponding system of differential equations.

(a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 4, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The solution to the system of DEs is then:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

SOLUTION: $\lambda^2 - 2\lambda + 5 = 0$, so $\lambda^2 - 2\lambda + 1 = -4$ and $\lambda = 1 \pm 2i$.
For $\lambda = 1 + 2i$, we have:

$$(3 - (1 + 2i))v_1 - 2v_2 = 0 \quad \Rightarrow \quad (2 - 2i)v_1 - 2v_2 = 0 \quad \Rightarrow \quad (1 - i)v_1 - v_2 = 0$$

so we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$. For $\lambda = 1 - 2i$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$, which is the complex conjugate (that always happens for real matrices A). To solve the corresponding system of differential equations, we first have to compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(1+2i)t} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = \mathrm{e}^t(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = \mathrm{e}^t \begin{bmatrix} \cos(2t) + i\sin(2t) \\ \cos(2t) + \sin(2t) + i(-\cos(2t) + \sin(2t)) \end{bmatrix}$$

The solution is then

$$\mathbf{x}(t) + C_1 \mathrm{Real}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \mathrm{Imag}(\mathrm{e}^{\lambda t}\mathbf{v})$$

which in this case is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^t \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + C_2 \mathrm{e}^t \begin{bmatrix} \sin(2t) \\ -\cos(2t) + \sin(2t) \end{bmatrix}$$

(c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = -3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = -1, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ And the solution to the corresponding system of DEs is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(d) $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 2+i, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$ and $\lambda_2 = 2-i, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1-i \end{bmatrix}$

For the solution, first compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(2+i)t} \begin{bmatrix} -2 \\ 1+i \end{bmatrix} = \mathrm{e}^{2t}(\cos(t) + i\sin(t)) \begin{bmatrix} -2 \\ 1+i \end{bmatrix} = \mathrm{e}^{2t} \begin{bmatrix} -2\cos(t) - 2i\sin(t) \\ \cos(t) - \sin(t) + i(\cos(t) + \sin(t)) \end{bmatrix}$$

The full solution is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} -2\cos(t) \\ \cos(t) - \sin(t) \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} -2\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}$$

3. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if A is given below:

(a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

SOLUTION: Substitute $x_2 = \frac{1}{2}(x_1' - x_1)$ into the second equation to get $x_1'' + 9x_1 = 0$, so

$$x_1(t) = C_1 \cos(3t) + c_2 \sin(3t)$$

Then put it back into the expression above to determine x_2:

$$x_2(t) = \frac{3C_2 - C_1}{2} \cos(3t) - \frac{3C_1 + C_2}{2} \sin(3t)$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

SOLUTION: Substitute $x_2 = x_1' - x_1$ into the second equation, and find that $x_1'' - 2x_1' - 3x_1 = 0$. From that,

$$x_1(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{3t} \qquad x_2(t) = -2C_1 \mathrm{e}^{-t} + 2C_2 \mathrm{e}^{3t}$$

(c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

SOLUTION: Note that its easier to start with the second equation rather than the first: $x_1 = -(x_2' + x_2)$. This gives us an IVP in x_2:

$$x_2'' - 2x_2' + x_2 = 0 \quad \Rightarrow \quad x_2(t) = \mathrm{e}^t(C_1 + C_2 t)$$

Then

$$x_1 = -2\mathrm{e}^t(C_1 + C_2 t) - C_2 \mathrm{e}^t$$