

Summary- Solve a Linear System

We started by looking at general systems of equations. Be able to convert an n^{th} order DE to a system of first order. Be able to convert a system of two first order equations to a single equation of second order. Be able to convert a system to dy/dx form.

Eigenvalues and Eigenvectors

For the following, we are solving the system:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \mathbf{x}' = A\mathbf{x}$$

1. Definition: If there is a constant λ and a non-zero vector \mathbf{v} that solves

$$\begin{aligned} (a - \lambda)v_1 + bv_2 &= 0 \\ cv_1 + (d - \lambda)v_2 &= 0 \end{aligned} \tag{1}$$

then λ is an eigenvalue, and \mathbf{v} is an associated eigenvector. This system has a non-zero solution for v_1, v_2 only if the two lines are multiples of each other. In that case, the determinant must be zero.

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \Rightarrow \lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$$

And this is the **characteristic equation**. This is formally solved via the quadratic formula, but we would typically factor it or complete the square. For each λ , we must go back and solve Equation (1) to find \mathbf{v} . As a shortcut, the eigenvector can be written down directly (as long as the equation is not $0 = 0$)

$$(a - \lambda)v_1 + cv_2 = 0 \Rightarrow \mathbf{v} = \begin{bmatrix} -c \\ a - \lambda \end{bmatrix}$$

Solve $\mathbf{x}' = A\mathbf{x}$

1. We make the ansatz: $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, substitute into the DE, and we find that λ, \mathbf{v} must be an eigenvalue, eigenvector of the matrix A .
2. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \quad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ :

- Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

- Complex $\lambda = a + ib$, \mathbf{v} (we only need one):

$$\mathbf{x}(t) = C_1 \operatorname{Re}(e^{\lambda t} \mathbf{v}) + C_2 \operatorname{Im}(e^{\lambda t} \mathbf{v})$$

- One eigenvalue, one eigenvector (which is not needed). Determine \mathbf{w} , where:

$$\begin{aligned} (a - \lambda)x_0 + cy_0 &= w_1 \\ cx_0 + (d - \lambda)y_0 &= w_2 \end{aligned}$$

Then

$$\mathbf{x}(t) = e^{\lambda t} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = e^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

Note: In this solution, we don't have arbitrary constants- rather, we're writing the solution in terms of the initial conditions.

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = e^{rt}$	$\mathbf{x} = e^{\lambda t} \mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\det(A - \lambda I) = 0$
Real Solns	$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
Complex	$y = C_1 \operatorname{Re}(e^{rt}) + C_2 \operatorname{Im}(e^{rt})$	$\mathbf{x}(t) = C_1 \operatorname{Re}(e^{\lambda t} \mathbf{v}) + C_2 \operatorname{Im}(e^{\lambda t} \mathbf{v})$
<i>SingleRoot</i>	$y = e^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = e^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$

Solve General Nonlinear Equations

We don't have a method that will work on every system of nonlinear differential equations, although there are some tricks we can try with special cases- that is, given the system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

And we might get lucky if it is in the form of an equation from Chapter 2.