Case 3: One Real Eigenvalue, One Eigenvector

The following example is a special case (not typical), but shows what might happen in the case that we have one repeated eigenvalue, AND two distinct eigenvectors. This case is actually treated as identical to "Case 1". Here's that special example:

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

Now, solve the system for $\mathbf{v}$:

$$\begin{aligned} (1-1)v_1 + 0v_2 &= 0 \\ 0v_1 + (1-1)v_2 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} 0v_1 + 0v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned}$$

Both v_1, v_2 are free variables, so any vectors would work- For example, here are two eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}(t) = \begin{bmatrix} C_1 \mathrm{e}^t \\ C_2 \mathrm{e}^t \end{bmatrix}$$

Typical Case: A double eigenvalue, one eigenvector

Example: $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ In this case, $\lambda = 2, 2$ but

$$\begin{aligned} 0v_1 + 3v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This one can be a little tricky, but there is an way to quickly get the solution if we have the intial conditions, $\mathbf{x}(0) = \mathbf{x}_0$. Then the solution to $\mathbf{x}' = A\mathbf{x}$ is given by:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

If we substitute this back into the DE, we will see that the following needs to hold:

$$\begin{aligned} (a-\lambda)x_0 + by_0 &= w_1 \\ cx_0 + (d-\lambda)y_0 &= w_2 \end{aligned} \qquad \text{or} \qquad (A - \lambda I)\mathbf{x}_0 = \mathbf{w}$$

Example:

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

We just computed the eigenvalues to be $\lambda = 2, 2$. To find the vector $\mathbf{w}$, we take:

$$\begin{aligned} (2-2)x_0 + 3y_0 &= w_1 \\ 0x_0 + (2-2)y_0 &= w_2 \end{aligned} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 3y_0 \\ 0 \end{bmatrix}$$

The full solution is then:

$$\mathbf{x}(t) = \mathrm{e}^{2t} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 3y_0 \\ 0 \end{bmatrix} \right)$$

Example:

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x}$$

The trace is 0 and the determinant is 0. Therefore, $\lambda = 0$ is the only eigenvalue. If there are no initial conditions, assume they are (x_0, y_0) as the last example. Then our vector $\mathbf{w}$ is computed as:

$$\begin{array}{ll} (4-0)x_0 - 2y_0 & = w_1 \\ 8x_0 - (4-0)y_0 & = w_2 \end{array} \qquad \mathbf{w} = \begin{bmatrix} 4x_0 - 2y_0 \\ 8x_0 - 4y_0 \end{bmatrix}$$

The solution is (in several forms):

$$\mathbf{x}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 4x_0 - 2y_0 \\ 8x_0 - 4y_0 \end{bmatrix}$$

We'll note that this is just a straight line in the (x_1, x_2) plane.

Summary

To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

  $$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

  $$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$ (not used directly).

  Use the initial condition, $\mathbf{x}_0 = (x_0, y_0)$ and the vector $\mathbf{w}$ so that

  $$\begin{array}{ll} (a-\lambda)x_0 + by_0 & = w_1 \\ cx_0 + (d-\lambda)y_0 & = w_2 \end{array} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{x}_0 = \mathbf{w}$$

  The solution is then

  $$\mathbf{x}(t) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

Homework: Linear Systems (Last Day!)

1. Solve $\mathbf{x}' = A\mathbf{x}$ for each matrix A below.

   (a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

   (d) $A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

2. For each matrix below, find the eigenvalues and eigenvectors (don't form the solution to a differential equation).

   (a) $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -1 \\ 3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

3. Given the eigenvector/eigenvalues, write the solution to $\mathbf{x}' = A\mathbf{x}$.

   (a) $\lambda_{1,2} = -2, 5$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (b) $\begin{bmatrix} -7 & 2 \\ -5 & -5 \end{bmatrix}, \lambda = -6 + 3i$ and $\mathbf{v}_1 = \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix}$

   (c) $\lambda_{1,2} = -1, -3$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$