Day 2 of Linear Systems (Replaces 7.2)

- What is a matrix? Determinant? Trace?
- Be able to convert a matrix-vector equation into a system of equations and vice versa.
- Be able to write the solution to the homogeneous system of equations in vector form (also- Equilibrium solutions)
- Recall how to write lines in 2d, 3d using vector format.
- Understand the vector $\mathrm{e}^{kt}\mathbf{v}$, especially geometrically.
- If ansatz if $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, what must be true for $\mathbf{x}' = A\mathbf{x}$?

Class Notes

First some useful notation.

Key Definition: A system of equations can be written in matrix-vector form as shown below (this is a definition)

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Similarly, we could extend this to three variables (this is just to show you what it would look like):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

More Definitions...

A **matrix** is simply an array of numbers, and the **size** of a matrix is defined as the number of *rows* $\times$ the number of *columns* (similar to a spreadsheet, rows always come first). We identify elements of the array by locating the row and column.

In the examples above, we have a 2×2 matrix and a 3×3 matrix. A vector can also be described as a 2×1 matrix (or 3×1).

The definitions above gives meaning to **matrix-vector multiplication**. For example,

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(2) \\ -1(1) + 3(2) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

In the previous lesson, we reviewed the **determinant**. The **trace** of a matrix is the sum of the diagonal elements of a matrix (from upper left element to the lower right element). For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = 4 - 6 = -2 \qquad \mathrm{Tr}(A) = 1 + 4 = 5$$

Review: Solving a System of Equations

In solving a linear system of equations, there are three (and only three) possible outcomes: (i) Exactly one solution (intersecting lines), (ii) No Solution (parallel lines), (iii) an infinite number of solutions (the same line).

We can solve the system using Cramer's Rule:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad \Rightarrow \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

This works as long as the determinant in the denominator is not zero. If that is the case, then we either have parallel lines or the same line.

Example

Solve the system:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

SOLUTION: The determinant is 0, and we see these are the same line. Therefore, any (x, y) satisfying $x + 2y = 0$ will solve the system- We want to represent the line in vector form.

Shortcut: The line $ax + by = 0$ can be represented by the vector $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -b \\ a \end{bmatrix}$

To solve the previous system, we should provide the line: $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Lines in Vector Form

Recall from Calculus III: A line in two or three dimensions can be defined if it goes through **point** p and goes in the **direction** q as (in parametric vector form):

$$\vec{p} + t\vec{q} \qquad -\infty < t < \infty$$

So, for example, the line going through the point $(1,2,3)$ in the direction of $\langle 1,-1,1\rangle$ can be written as:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+t \\ 2-t \\ 3+t \end{bmatrix}$$

Class example: What does this look like:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mathrm{e}^t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad -\infty < t < \infty$$

SOLUTION: It is a ray rather than a line. As $t \to -\infty$, $\mathrm{e}^t \to 0$ (the line goes to (1,2,3)), then as t increases, we move farther and farther in the direction given.

Systems of DEs and Matrices

Definition: An **autonomous** system of first order **linear** differential equations is a system of the following form- All 3 forms of the notation are used at some point:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

(Note: Autonomous means that t is not explicitly part of the DEs)
Definition: A **solution** to the system is a parametric function that satisfies the given relationship.

To find the **equlibrium solution**, set the derivatives equal to zero and solve the corresponding system- you might note that the zero vector is always an equilibrium solution, but there may be more.

Example

Show that $\mathbf{x}(t) = \mathrm{e}^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ solves the system:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the derivatives, remember that $\mathbf{x}(t)$ is in parametric form, where $x_1(t) = \mathrm{e}^{-t}$ and $x_2(t) = 2\mathrm{e}^{-t}$. We differentiate component-wise, so

$$\mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-t} \\ -2\mathrm{e}^{-t} \end{bmatrix} = -\mathrm{e}^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(Note that we could have simply differentiated the exponent in front, since that was the only term that involved t).

Now we want to see if this is the same as $A\mathbf{x}$. Since e^{-t} is a scalar, I'm able to factor it out:

$$A\mathbf{x} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = e^{-t} \begin{bmatrix} 3-4 \\ 2-4 \end{bmatrix} = e^{-t} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We see that indeed, $\mathbf{x}'(t)$ is the same as $A\mathbf{x}(t)$.

Towards a General Solution

Suppose we have the system of differential equations:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned}$$

with the ansatz below, where we stress that $\mathbf{v} \neq 0$:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(Note that λ, v_1, v_2 are all unknown in the ansatz as of now). What must be true of λ, v_1, v_2?

Let's substitute our solution into the system and see what we get. First, the derivative is simple:

$$\mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \lambda e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

And putting this into the system and simplifying:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda e^{\lambda t} v_1 &= a e^{\lambda t} v_1 + b e^{\lambda t} v_2 \\ \lambda e^{\lambda t} v_2 &= c e^{\lambda t} v_1 + d e^{\lambda t} v_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} av_1 + bv_2 &= \lambda v_1 \\ cv_1 + dv_2 &= \lambda v_2 \end{aligned}$$

From this, we get the really important set of equations that we want to remember. In order for λ and $\mathbf{v}$ to be in the solution $\mathbf{x}(t)$, we need:

$$\begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array}$$

Now, if we were solving this using Cramer's Rule, the determinant

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

would be in the denominator, and if this were non-zero, the ONLY SOLUTION is the zero vector. We don't want that- We want $\mathbf{v}$ to be NOT zero, so we must find λ so that the determinant IS zero. That is our first condition: Find λ so that

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

Once we find λ, we go back to the system we were looking at, and substitute our new value of λ into the system:

$$\begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & =0 \\ cv_1 & +(d-\lambda)v_2 & =0 \end{array}$$

Which is guaranteed to represent two multiples of the same line (that's how we defined λ). We'll continue this discussion in our next lecture.

Homework (to replace 7.2)

1. What will the graph of $e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be (where t is any real number).

2. Verify that $\mathbf{x}_1(t)$ below satisfies the DE below.

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3. Each system below is *nonlinear*. Solve each by first writing the system as dy/dx.

   (a) $\begin{array}{ll} x' & = y(1+x^3) \\ y' & = x^2 \end{array}$ (b) $\begin{array}{ll} x' & = 4+y^3 \\ y' & = 4x-x^3 \end{array}$ (c) $\begin{array}{ll} x' & = 2x^2y+2x \\ y' & = -(2xy^2+2y) \end{array}$

   (Note: Some of these may be **exact**.)

4. For each matrix below, compute the determinant $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$. Your determinant should be an expression in λ.

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

5. Show that if the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$ can be expressed as:

$$\lambda^2 - tr(A)\lambda + \det(A)$$

   where $\mathrm{tr}(A)$ is the trace of A and $\det(A)$ is the determinant of A.

6. Suppose $\lambda = 2$. Using the matrix $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$, solve the following system for v_1, v_2. Be sure to write your answer in vector form.

$$\begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & =0 \\ cv_1 & +(d-\lambda)v_2 & =0 \end{array}$$

7. Suppose $\lambda = -1$. Using the matrix $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$, solve the following system for v_1, v_2. Be sure to write your answer in vector form.

$$\begin{aligned} (a-\lambda)v_1 \quad & +bv_2 &= 0 \\ cv_1 \quad & +(d-\lambda)v_2 &= 0 \end{aligned}$$