Lecture Notes to substitute for 7.3-7.5

We want to solve the system:

$$\begin{array}{ll} x_1' & = ax_1 + bx_2 \\ x_2' & = cx_1 + dx_2 \end{array} \quad \Rightarrow \quad \mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

SOLUTION: Use the ansatz $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$.

Then $\mathbf{x}' = \lambda \mathrm{e}^{\lambda t}\mathbf{v}$, so that the DE becomes:

$$A\mathrm{e}^{\lambda t}\mathbf{v} = \lambda \mathrm{e}^{\lambda t}\mathbf{v} \quad \Rightarrow A\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \begin{array}{l} av_1 + bv_2 = \lambda v_1 \\ cv_2 + dv_2 = \lambda v_2 \end{array}$$

If the system above is true for that particular value of λ and **non-zero** vector $\mathbf{v}$, then λ is an **eigenvalue** of the matrix A and $\mathbf{v}$ is an associated **eigenvector**. Note that while $\mathbf{v}$ is not allowed to be the zero vector, λ could be zero.

Computing Eigenvalues and Eigenvectors:

Consider the system we had:

$$\begin{array}{ll} av_1 + bv_2 & = \lambda v_1 \\ cv_1 + dv_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array}$$

That is the key system of equations. We saw last time $A\mathbf{x} = 0$ has exactly the zero solution iff $\det(A) \neq 0$. Therefore, for this system to have a **non-trivial solution** (which is a non-zero eigenvector), the **determinant must be zero**.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

You might recognize those two quantities that are computed as the trace and determinant of A:

$$\mathrm{Tr}(A) = a + d \qquad \det(A) = ad - bc$$

Theorem: The eigenvalues for the 2×2 matrix A are found by solving the **characteristic equation**:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

So, given A, compute the $\mathrm{Tr}(A)$, the $\det(A)$ and the discriminant,

$$\Delta = (\mathrm{Tr}(A))^2 - 4\det(A)$$

Then the eigenvalues are:

$$\lambda = \frac{Tr(A) \pm \sqrt{\Delta}}{2}$$

Once we have computed λ, we go back to the system of equations and solve for v_1, v_2.

Examples: Solve for eigenvalues and eigenvectors

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find eigenvalues and eigenvectors for A.

SOLUTION: We could jump right to the characteristic equation, but for practice its good to write down what it is we actually want to solve (the unknowns below are λ, v_1, v_2):

$$\begin{aligned} (7-\lambda)v_1 + 2v_2 &= 0 \\ -4v_2 + (1-\lambda)v_2 &= 0 \end{aligned}$$

For this to have a non-zero solution v_1, v_2, the determinant must be zero:

$$(7-\lambda)(1-\lambda) + 8 = 0 \quad \Rightarrow \quad \lambda^2 - 8\lambda + 15 = 0$$

(Note that the trace is 8, determinant is 15). This factors, so we can solve for λ:

$$(\lambda - 5)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 3, 5$$

Now, for each λ, go back to our system of equations for v_1, v_2 and solve. NOTE: By design, these equations should be multiples of each other!

For $\lambda = 3$:

$$\begin{aligned} (7-3)v_1 + 2v_2 &= 0 \\ -4v_2 + (1-3)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad 4v_1 + 2v_2 = 0 \quad \text{or} \quad 2v_1 + v_2 = 0$$

There are an infinite number of solutions (and there always be). We want to choose "nice" values of v_1, v_2 that satisfies this relationship (or alternatively, lies on the line). An easy way of writing v_1, v_2 is to notice the following:

Given $ax + by = 0$, we can choose $x = b, y = -a$ to lie on the line.

Continuing, we'll take our vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Before we continue to the next eigenvalue, you might notice that any constant multiple of an eigenvector is an eigenvector. Therefore, we could have chosen any of the following:

$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}, \text{ etc.}$$

Now go through the same process to find the eigenvector for $\lambda = 5$:

$$\begin{aligned} (7-5)v_1 + 2v_2 &= 0 \\ -4v_1 + (1-5)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad 2v_1 + 2v_2 = 0 \quad \Rightarrow \quad v_1 + v_2 = 0$$

Therefore, we take $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For the full solution, we take a linear combination just as we did in Chapter 3. Here, the full general solution is

$$\mathbf{x}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_1 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example: Solve a Linear System

Solve the linear system using eigenvectors:

$$\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$$

SOLUTION: We find the eigenvalues λ_1, λ_2, and the corresponding eigenvectors. Then the solution is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t} \mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t} \mathbf{v}_2$$

With that, we first compute the eigenvalues and eigenvectors. The determinant is 8, the trace is 6. The characteristic equation is:

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 4) = 0$$

Therefore, $\lambda = 2, 4$.

- For $\lambda = 2$:

$$\begin{aligned} (3-2)v_1 + v_2 &= 0 \\ v_1 + (3-2)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} v_1 + v_2 &= 0 \\ v_1 + v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- For $\lambda = 4$,

$$\begin{aligned} (3-4)v_1 + v_2 &= 0 \\ v_1 + (3-4)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -v_1 + v_2 &= 0 \\ v_1 - v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example: Solve the linear system.

The technique is the same as the last example. Write it down, try it out, then come back to this page to see if we have the same answer.

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} \qquad \mathrm{Tr}(A) = 1 \quad \det(A) = -2 \quad \Delta = 9$$

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, or $(\lambda + 1)(\lambda - 2) = 0$.

The eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found by solving the system above. For $\lambda = -1$:

$$\begin{aligned} (3+1)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2+1)v_2 &= 0 \end{aligned} \qquad 2v_1 - v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{aligned} (3-2)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2-2)v_2 &= 0 \end{aligned} \qquad v_1 - 2v_2 = 0 \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution to the system of differential equations is then:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case 2: Complex Eigenvalues

First, let's look at the eigenvalue/eigenvector computations themselves in an example: Find the eigenvalues and eigenvectors for the matrix below:

$$\begin{array}{rl} 3v_1 - 2v_2 & = \lambda v_1 \\ v_1 + v_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rl} (3-\lambda)v_1 - 2v_2 & = 0 \\ v_1 + (1-\lambda)v_2 & = 0 \end{array}$$

SOLUTION: Form the characteristic equation using the shortcut or by taking the determinant of the coefficient matrix:

$$\lambda^2 - Tr(A)\lambda + \det(A) = 0 \qquad \lambda^2 - 4\lambda + 5 = 0 \qquad \lambda = 2 \pm i$$

Now, if $\lambda = 2 + i$, solve for an eigenvector:

$$\begin{array}{rrl} (3-(2+i))v_1 & -2v_2 & = 0 \\ v_1 & +(1-(2+i))v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (1-i)v_1 & -2v_2 & = 0 \\ v_1 & +(-1-i)v_2 & = 0 \end{array}$$

Side Note/Side Computation

Recall that we said that these equations needed to be the same line- Indeed they are. To see this, if you divide the first equation by $1 - i$, we get:

$$\frac{1-i}{1-i}v_1 - \frac{2}{1-i}v_2 = 0 \quad \Rightarrow \quad v_1 - \frac{2(1+i)}{(1^2+1^2)}v_2 = 0 \quad \Rightarrow v_1 - (1+i)v_2 = 0$$

which is the second equation.

Returning to the Problem...

Given $(1-i)v_1 - 2v_2 = 0$, we can use $\mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$.

As a side remark, the other eigenvalue/eigenvector pair are the complex conjugates (we won't be using them):

$$\lambda_2 = 2 - i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

The next section tells us how to solve the system.

Applying Complex evals to Systems of DEs

Suppose we have a complex eigenvalue, $\lambda = a \pm ib$. Use one of them to construct the corresponding eigenvector (complex) $\mathbf{v}$. We can then solve the system using the theorem below.

Theorem: Given $\lambda = a + ib$, $\mathbf{v}$ for a matrix A in $\mathbf{x}' = A\mathbf{x}$, the solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \mathrm{Re}\left(e^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(e^{\lambda t}\mathbf{v}\right)$$

Notice that this is the extension of what we did in Chapter 3.

Example

Give the general solution to the system $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{x}$

This is the system for which we already have the eigenvalues and eigenvectors:

$$\lambda = 2 + i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 - i \end{bmatrix}$$

Now, compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(2+i)t} \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = \mathrm{e}^{2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} =$$

$$\mathrm{e}^{2t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ (\cos(t) + \sin(t)) + i(-\cos(t) + \sin(t)) \end{bmatrix}$$

so that the general solution is given by:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix} + C_1 \mathrm{e}^{2t} \begin{bmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{bmatrix}$$

Geometrically, the origin is a *spiral source*. As a side remark, if I had solved the second equation for x_1 and substituted it into the first, I would have had:

$$x_2'' - 4x_2' + 5x_2 = 0 \quad \Rightarrow \quad r = 2 \pm i \quad \Rightarrow \quad x_2 = C_1 \mathrm{e}^{2t}\cos(t) + C_2 \mathrm{e}^{2t}\sin(t)$$

Example

Give the general solution to the system: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

First, the characteristic equation: $\lambda^2 + 1 = 0$, so that $\lambda = \pm i$.

Now we solve for the eigenvector to $\lambda = i$:

$$\begin{aligned} (2 - i)v_1 - 5v_2 &= 0 \\ 1v_1 + (-2 - i)v_2 &= 0 \end{aligned}$$

Using the second equation, $v_1 - (2 + i)v_2 = 0$, and we have our eigenvalue/eigenvector pair. Now we compute the needed quantity, $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{it} \begin{bmatrix} 2 + i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t)) \begin{bmatrix} 2 + i \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos(t) + i\sin(t))(2 + i) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

Simplifying, we get:

$$\begin{bmatrix} (2\cos(t) - \sin(t)) + i(2\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

The solution is:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

We will quickly verify that this is what we would get using the techniques of Chapter 3. From the second equation, solve for x_1, then use the first equation to get a second order DE for x_2.

$$x_1 = x_2' + 2x_2 \quad \Rightarrow \quad (x_2'' + 2x_2') = 2(x_2' + 2x_2) - 5x_2 \quad \Rightarrow \quad x_2'' + x_2 = 0$$

Therefore, $x_2 = C_1\cos(t) + C_2\sin(t)$. Solving for x_1:

$$x_1 = x_2' + 2x_2 = (-C_1\sin(t) + C_2\cos(t)) + 2(C_1\cos(t) + C_2\sin(t))$$

and we see that we get the identical solution.

Graphically, the solutions are ellipses. In fact, if we solve the differential equation by computing dy/dx, we get solutions of the form:

$$x^2 - 4xy + 5y^2 = C$$

Next time, we'll finish the solutions by looking at the case of a single eigenvalue.

Exercise Set, Day 3 (HW to replace 7.3, 7.5)

This homework is all about solving for eigenvalues and eigenvectors, and we'll also do some visualization and classification of equilibria.

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

2. For each matrix, first find the eigenvalues and eigenvectors, then solve the corresponding linear system of differential equations.

(a) $\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

3. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if A is given below:

(a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$