Outline- Section 3.8

- Full model: $mu'' + \gamma u' + ku = F(t)$.
- Last time: Generic $F(t)$
  (Method of Undet Coeffs or Var of Params)
- This time: $F(t) = \cos(\omega t)$ or $\sin(\omega t)$ (Periodic forcing).

Side remark on Vocab:

1. **Transient part of the solution** part of soln $\rightarrow$ *zero* as $t \rightarrow \infty$.
   $m, \gamma, k > 0$ then $y_h(t) \rightarrow 0$.
2. **Forced Response** part of soln remaining.
   Also called the steady state solution.

Today we look at the forced response...

But what's new

Analyze in the case of our model:

- No Damping, Periodic Forcing.
- Damping and Periodic Forcing.

No Damping, Periodic Forcing

Model

$$mu'' + ku = A\cos(\omega t) \quad \Rightarrow \quad u'' + \omega_0^2 u = F_0 \cos(\omega t)$$

The homogeneous part of the solution can be expressed as:

$$u_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

For the particular solution, two subcases: $\omega \neq \omega_0$ and $\omega = \omega_0$

Forcing and Natural Frequencies are Distinct

$$u'' + \omega_0^2 u = F_0 \cos(\omega t)$$

If $\omega_0 \neq \omega$,then ansatz:
$y_p = A\mathrm{e}^{i\omega t}$ and $y_p' = i\omega A\mathrm{e}^{i\omega t}$and $y_p'' = -\omega^2 A\mathrm{e}^{i\omega t}$.

$$A\mathrm{e}^{i\omega t}(-\omega^2 + \omega_0^2) = F_0\mathrm{e}^{i\omega t} \quad \Rightarrow \quad A = \frac{F_0}{\omega_0^2 - \omega^2}$$

so the overall solution is:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2} \cos(\omega t)$$

To simplify the analysis, we'll also assume $y(0) = y'(0) = 0$ (Exercise 1)

$$C_1 = -\frac{F_0}{\omega_0^2 - \omega^2} \qquad C_2 = 0$$

In this case,

$$y(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

To "simplify" our analysis, we rewrite this difference as a product.

Write solution as a product

Using some trig (See pg 213):

$$\frac{2F_0}{\omega_0^2 - \omega^2} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right) \sin\left(\frac{(\omega_0 + \omega)t}{2}\right)$$

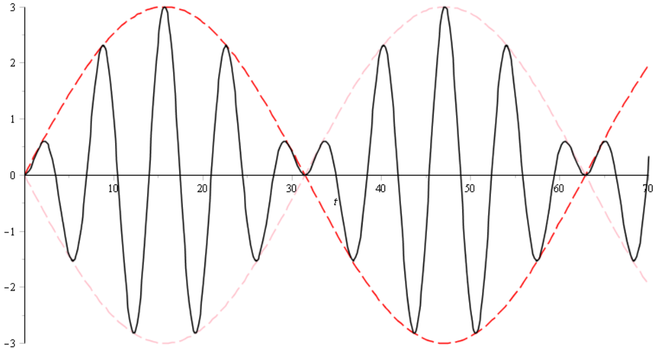
If $\omega \approx \omega_0$, the first term is a larger and longer wave.

$$\pm \frac{2F_0}{\omega_0^2 - \omega^2} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)$$

Let's see a graph...

Graph of the solution, $\omega = 0.8, \omega_0 = 1$

```
A:=plot(3*sin((1/10)*t),t=0..70,linestyle=dash,color=red):
B:=plot(-3*sin((1/10)*t),t=0..70,linestyle=dash,color=pink):
C:=plot(3*sin((1/10)*t)*sin((9/10)*t),t=0..70,linestyle=solid,color=black):
display({A,B,C});
```



This is **Beating...**
Circular frequency of a single beat (half the frequency of the sine): $|\omega_0 - \omega|$.

What happens as $\omega \to \omega_0$?

$$\frac{2F_0}{\omega_0^2 - \omega^2} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right) \sin\left(\frac{(\omega_0 + \omega)t}{2}\right)$$

If $\omega \approx \omega_0$, the first term is a larger and longer wave.

$$\pm \frac{2F_0}{\omega_0^2 - \omega^2} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)$$

The "envelope" gets longer and longer with larger and larger amplitude....

What happens at $\omega = \omega_0$? Something known as **resonance**.

Several ways of getting the particular solution:

1. Start Method of Undet Coeffs over again.
   Ansatz: $u_p = At\mathrm{e}^{i\omega_0 t}$.
2. Use the previous solution, and take the limit as $\omega \to \omega_0$.

Take limit (l'Hospital's Rule)

$$\lim_{\omega \to \omega_0} \frac{F_0 \left(\cos(\omega t) - \cos(\omega_0 t)\right)}{\omega_0^2 - \omega^2} =$$

$$\lim_{\omega \to \omega_0} \frac{-F_0 t \sin(\omega t)}{-2\omega} = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

Now "blows up", or becomes unbounded... This is resonance.

Resonance occurs when the forcing freq matches the natural freq.

Summary so far...

Given $u'' + \omega_0^2 u = F_0 \cos(\omega t)$

1. If $\omega \approx \omega_0$, we get **Beating**.
   The (circ) freq of one beat is $|\omega_0 - \omega|$.
2. If $\omega = \omega_0$, then **Resonance**

(Pause for the video)

Full model, Periodic Forcing

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

Slightly altered:

$$u'' + pu' + qu = \cos(\omega t)$$

The characteristic equation has roots:

$$r = \frac{-p \pm \sqrt{p^2 - 4q}}{2}, \quad p > 0$$

Implies that, always, we have

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

No solution is completely unbounded...

The Big Question

Does a form of resonance persist?

Given a fixed p, q in the model, is it possible to **tune** the forcing frequency to **maximize** the amplitude of the forced response?

Answer: It is! (Break a wine glass with your voice)

Full model, Periodic Forcing

$$u'' + pu' + qu = \cos(\omega t)$$

Ansatz $y_p = A\mathrm{e}^{i\omega t}$ and $y_p' = Ai\omega\mathrm{e}^{i\omega t}$ and $y_p'' = -A\omega^2\mathrm{e}^{i\omega t}$.
Then

$$A\mathrm{e}^{i\omega t}(-\omega^2 + i\omega p + q) = \mathrm{e}^{i\omega t}$$

so that

$$A = \frac{1}{(q - \omega^2) + i\omega p}$$

and the particular solution is

$$u_p = \mathrm{Re}\left(A\mathrm{e}^{i\omega t}\right)$$

Given

$$u_p(t) = \mathrm{Real}\left(\frac{1}{(q-\omega^2)+i\omega p}\mathrm{e}^{i\omega t}\right) = R\cos(\omega t - \delta)$$

(See Complexification Handout) Amplitude R and phase angle δ for u_p are given by:

$$R(\omega) = \frac{1}{|(q-\omega^2)+i\omega p|} \qquad \delta = \tan^{-1}\left(\frac{\omega p}{q-\omega^2}\right)$$

We observe that R (the amplitude of u_p) is a function of ω.

Can we maximize R?

Set the derivative to zero and solve for ω...
NOTE: If

$$R = \frac{1}{\sqrt{f(\omega)}} \quad \Rightarrow \quad R' = \frac{1}{2}(f(\omega))^{-1/2} f'(\omega)$$

Therefore, if we solve for $R' = 0$, we only need $f'(\omega) = 0$

Continuing - Where is R at its maximum?

$$R = \frac{1}{|(q-\omega^2)+i\omega p|} = \frac{1}{\sqrt{(q-\omega^2)^2+\omega^2 p^2}}$$

$$f(\omega) = (q-\omega^2)^2 + p^2\omega^2 \quad \Rightarrow \quad \frac{df}{d\omega} = 2(q-\omega^2)(-2\omega) + p^2 \cdot 2\omega = 0$$

Solving for only the positive ω, we get $\omega = \sqrt{\frac{2q-p^2}{2}}$.

Numerical Example

Find the forced response to

$$u'' + u' + 2u = \cos(2t)$$

SOLUTION: Ansatz is $A\mathrm{e}^{2it}$.Substitute and factor:

$$A\mathrm{e}^{2it}(-4 + 2i + 2) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-2 + 2i}$$

Therefore, the amplitude and phase will be

$$R = \frac{1}{|-2+2i|} = \frac{1}{\sqrt{4+4}} = \frac{1}{\sqrt{8}} \qquad \delta = \arg(-2+2i) = \frac{3\pi}{4}$$

Therefore,

$$u_p = \frac{1}{2\sqrt{2}} \cos\left(2t - \frac{3\pi}{4}\right)$$

Numerical Example 2

Find ω that maximizes the amplitude of the forced response to:

$$u'' + u' + 2u = \cos(\omega t)$$

SOLUTION: Ansatz is $Ae^{i\omega t}$. Substitute and factor:

$$Ae^{i\omega t}(-\omega^2 + i\omega + 2) = e^{i\omega t} \quad \Rightarrow \quad A = \frac{1}{(2-\omega^2) + i\omega}$$

Therefore, the amplitude is

$$R = \frac{1}{|(2-\omega^2) + i\omega|} = \frac{1}{\sqrt{(2-\omega^2)^2 + \omega^2}} = \frac{1}{\sqrt{f(\omega)}}$$

Therefore, $R'(\omega) = 0$ when $f'(\omega) = 0$,which is computed:

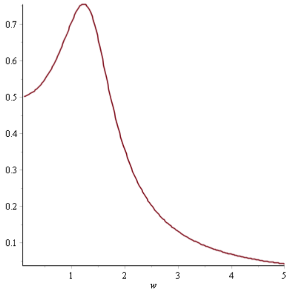
$$f(\omega) = (2-\omega^2)^2 - \omega^2 \quad \Rightarrow \quad f'(\omega) = 2(2-\omega^2)(-2\omega) + 2\omega = 0$$

Solving for ω, we get $\omega = \sqrt{\frac{3}{2}}$

```
> with(plots):
> R:=1/(sqrt( (2-w^2)^2 + w^2 ));
```

$$R := \frac{1}{\sqrt{\left(-w^2+2\right)^2+w^2}}$$

```
> plot(R,w=0.1..5);
```




```
> solve(diff(R,w)=0,w);
```

$$0,\ \frac{1}{2}\sqrt{6},\ -\frac{1}{2}\sqrt{6}$$

Numerical Example 3

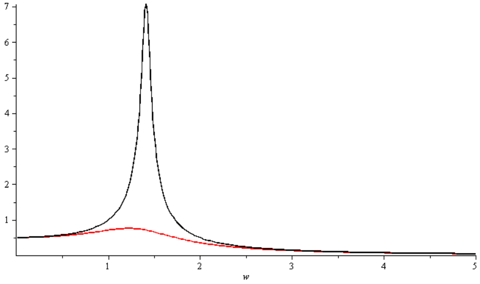
Find ω that maximizes the amplitude of the forced response to:

$$u'' + \frac{1}{10}u' + 2u = \cos(\omega t)$$

We'll just compare the maximum amplitude:

$$R = \frac{1}{|(2-\omega^2) + i\omega/10|} = \frac{1}{\sqrt{(2-\omega^2)^2 + \omega^2/100}}$$

(Red- Earlier R, Black- The R shown here)



What have we shown?

- With damping and periodic forcing, no unbdd solns.
- **However,** are able to "tune" the freq of the forcing function to maximize the response.
- The smaller the relative size of the damping, the larger the maximum amplitude.
- This has very important engineering implications
  (Go to video!)