Given $y'' + p(t)y' + q(t)y = g(t)$ and y_1, y_2 as y_h.Ansatz:

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

We will see that u_1, u_2 satisfy

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t) \end{aligned}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

Proof of Variation of Parameters

The steps for the algorithm:

- Step 1: Differentiate y_p to get y_p'.
- Step 2: Assume that $u_1'y_1 + u_2'y_2 = 0$.
- Step 3: Differentiate y_p again to get y_p''
- Step 4: Substitute y_p into the DE to see if we can solve for u_1, u_2.

Steps 1 and 2: Differentiate, get one constraint

$$y_p = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Constraint 1: $u_1'y_1 + u_2'y_2 = 0$. That leaves us:

$$y_p' = u_1y_1' + u_2y_2'$$

Differentiate again:

$$y_p'' = u_1'y_1 + u_1y_1'' + u_2'y_2 + u_2y_2''$$

And substitute into the DE.

Step 4: Substitution and Constraint 2

$$\begin{array}{rlllll} y_p'' & = u_1'y_1 & +u_1y_1'' & +u_2'y_2 & +u_2y_2'' \\ +py_p' & = & +pu_1y_1' & & +pu_2y_2' \\ +qy_p & = & +qu_1y_1 & & +qu_2y_2 \\ \hline g(t) & = u_1'y_1 & & +u_2'y_2 & \end{array}$$

where $u_1(y_1'' + py_1' + qy_1) = 0$ and $u_2(y_2'' + py_2' + qy_2) = 0$ Therefore, u_1, u_2 satisfy

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t) \end{aligned}$$

Conclusion

Variation of Parameters is a method to obtain the particular part of the solution, y_p, to a second order linear differential equation,

$$y'' + p(t)y' + q(t)y = g(t)$$

where we first find $y_h = C_1 y_1 + C_2 y_2$, then take as our ansatz:

$$y_p = u_1 y_1 + u_2 y_2$$

Then, u_1, u_2 satisfy the following equations, which can be solved by Cramer's Rule.

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

Example 1

Find y_p if $4y'' - 4y' - 8y = 8\mathrm{e}^{-t}$.

- Standard form to get the right $g(t)$. $y'' - y' - 2y = 2\mathrm{e}^{-t}$.
- $y_1 = \mathrm{e}^{2t}, y_2 = \mathrm{e}^{-t}$ and $g(t) = 2\mathrm{e}^{-t}$
- Solve the system- First, the denominators are $W(y_1, y_2)$.

$$W = \begin{vmatrix} \mathrm{e}^{2t} & \mathrm{e}^{-t} \\ 2\mathrm{e}^{2t} & -\mathrm{e}^{-t} \end{vmatrix} = -3\mathrm{e}^{t}$$

Example 1, continued

$$u_1' = \frac{\begin{vmatrix} 0 & \mathrm{e}^{-t} \\ 2\mathrm{e}^{-t} & -\mathrm{e}^{-t} \end{vmatrix}}{-3\mathrm{e}^{t}} = \frac{-2\mathrm{e}^{-2t}}{-3\mathrm{e}^{t}} = \frac{2}{3}\mathrm{e}^{-3t} \quad \Rightarrow \quad u_1(t) = -\frac{2}{9}\mathrm{e}^{-3t}$$

$$u_2' = \frac{\begin{vmatrix} \mathrm{e}^{2t} & 0 \\ 2\mathrm{e}^{2t} & 2\mathrm{e}^{-t} \end{vmatrix}}{-3\mathrm{e}^{t}} = \frac{2\mathrm{e}^{t}}{-3\mathrm{e}^{t}} = -\frac{2}{3} \quad \Rightarrow \quad u_2(t) = -\frac{2}{3}t$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{2}{9}\mathrm{e}^{-3t}\mathrm{e}^{2t} - \frac{2}{3}t\mathrm{e}^{-t} = -\frac{2}{9}\mathrm{e}^{-t} - \frac{2}{3}t\mathrm{e}^{-t}$$

We can take

$$y_p = -\frac{2}{3}t\mathrm{e}^{-t}$$

Do I need to take the constant when integrating u_1, u_2? If we do, then we would have:

$$u_1(t) + C_1, \qquad u_2(t) + C_2$$

so that

$$y_p = (u_1(t) + C_1)y_1 + (u_2(t) + C_2)y_2 = u_1 y_1 + u_2 y_2 + (C_1 y_1 + C_2 y_2)$$

Answer:

No, since we'll just be attaching part of the homogeneous solution.

Example 2

Use any method to find the general solution:

$$y'' - 2y' + y = \frac{\mathrm{e}^t}{1+t^2} + 3\mathrm{e}^t$$

SOLUTION: Break into three parts.

- $y'' - 2y' + y = 0$ which is $y(t) = \mathrm{e}^t(C_1 + C_2 t)$
- $y'' - 2y' + y = 3\mathrm{e}^t$ by Method of Undet Coeffs
- $y'' - 2y' + y = \frac{\mathrm{e}^t}{1+t^2}$ by Var of Params.

We'll focus on the last one.

Computations

For Variation of Parameters, we take

$$y_1 = \mathrm{e}^t \qquad y_2 = t\mathrm{e}^t \qquad g(t) = \frac{\mathrm{e}^t}{1+t^2}$$

We'll need $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \mathrm{e}^t & t\mathrm{e}^t \\ \mathrm{e}^t & (1+t)\mathrm{e}^t \end{vmatrix} = (1+t)\mathrm{e}^{2t} - t\mathrm{e}^{2t} = \mathrm{e}^{2t}$$

Solve for $u_1(t), u_2(t)$

$$u_1' = \frac{-y_2 g(t)}{W(y_1, y_2)} = \frac{-t\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{-t}{1+t^2} \quad \Rightarrow \quad u_1 = -\frac{1}{2}\ln(1+t^2)$$

$$u_2' = \frac{y_1 g(t)}{W(y_1, y_2)} = \frac{\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{1}{1+t^2} \quad \Rightarrow \quad u_2 = \tan^{-1}(t)$$

The particular solution for $g_2(t) = \frac{\mathrm{e}^t}{1+t^2}$ is given by:

$$y_p(t) = -\frac{1}{2}\mathrm{e}^t \ln(1+t^2) + t\mathrm{e}^t \tan^{-1}(t)$$

Full solution

The full solution to:

$$y'' - 2y' + y = \frac{\mathrm{e}^t}{1+t^2} + 3\mathrm{e}^t$$

is

$$y(t) = \mathrm{e}^t\left(C_1 + C_2 t + \frac{3}{2}t^2\right) - \frac{1}{2}\mathrm{e}^t \ln(1+t^2) + t\mathrm{e}^t \tan^{-1}(t)$$

Use the Variation of Parameters to solve

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3$$

Where we're given:

$$y_1(t) = t \quad y_2(t) = t\mathrm{e}^t$$

SOLUTION

Put in standard form first!

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t$$

Therefore,

$$g(t) = 2t \qquad W(y_1, y_2) = t^2 \mathrm{e}^t$$

And solve:

$$u_1' = \frac{-2t^2\mathrm{e}^t}{t^2\mathrm{e}^t} = -2 \quad \Rightarrow \quad u_1 = -2t$$

Similarly,

$$u_2' = \frac{2t^2}{t^2\mathrm{e}^t} = 2\mathrm{e}^{-t} \quad \Rightarrow \quad u_2 = -2\mathrm{e}^{-t}$$

so

$$y_p = (-2t)(t) + (-2\mathrm{e}^{-t})(t\mathrm{e}^t) = -2t^2 - 2t \quad \Rightarrow y_p(t) = -2t^2$$

Summary of Techniques - Chapter 3

1. One method to find y_h given $ay'' + by' + cy = 0$.
2. Two methods to find y_2 to $y'' + p(t)y' + q(t)y = 0$.
   (Must have y_1 already)
   - Reduction of Order: Assume $y_2 = v(t)y_1(t)$.
   - Wronskian: Compute the Wronskian two ways-
     Abel's Theorem and the def.
3. Two methods to find $y_p(t)$ for $y'' + p(t)y' + q(t)y = g(t)$.
   - Method of Undetermined Coefficients
     ($g(t)$ has to be of a certain type).
   - Variation of Parameters