

Exam 2 Summary

The exam will cover material from Section 3.1 to 3.8 except for 3.6 (Variation of Parameters). Here is a summary of that information.

Existence and Uniqueness:

Given the second order linear IVP,

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0$$

If there is an open interval I on which p, q , and g are continuous and contain t_0 , then there exists a unique solution to the IVP, valid on I (and may contain the endpoints of I , if the functions are also continuous there).

Structure and Theory (Mostly 3.2)

The goal of the theory was to establish the structure of solutions to the second order IVP:

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0$$

We saw that two functions form a **fundamental set** of solutions to the homogeneous DE if the **Wronskian** is not zero at t_0 .

1. Vocabulary: Linear operator, general solution, fundamental set of solutions, linear combination.
2. Theorems:

- Abel's Theorem.

If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian, $W(y_1, y_2)$, is either always zero or never zero on the interval for which the solutions are valid.

That is because the Wronskian may be computed as:

$$W(y_1, y_2)(t) = Ce^{-\int p(t) dt}$$

- The Structure of Solutions to $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0, y'(t_0) = v_0$

Given a fundamental set of solutions to the homogeneous equation, y_1, y_2 , then there is a solution to the initial value problem, written as:

$$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$$

where $y_p(t)$ solves the non-homogeneous equation.

In fact, if we have: $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \dots + g_n(t)$, we can solve by splitting the problem up into smaller problems:

- y_1, y_2 form a fundamental set of solutions to the homogeneous equation.
- y_{p_1} solves $y'' + p(t)y' + q(t)y = g_1(t)$
- y_{p_2} solves $y'' + p(t)y' + q(t)y = g_2(t)$
and so on..
- y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$

and the full solution is: $y(t) = C_1y_1 + C_2y_2 + y_{p_1} + y_{p_2} + \dots + y_{p_n}$.

Finding the Homogeneous Solution

We had two distinct equations to solve-

$$ay'' + by' + cy = 0 \quad \text{or} \quad y'' + p(t)y' + q(t)y = 0$$

First we look at the case with constant coefficients, then we look at the more general case.

Constant Coefficients

To solve

$$ay'' + by' + cy = 0$$

we use the **ansatz** $y = e^{rt}$. Then we form the associated **characteristic equation**:

$$ar^2 + br + c = 0 \quad \Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way:

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2 . The general solution is:

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

If $a, b, c > 0$ (as in the Spring-Mass model) we can further say that r_1, r_2 are negative. We would say that this system is **OVERDAMPED**.

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

$$y_h(t) = e^{-(b/2a)t} (C_1 + C_2 t)$$

If $a, b, c > 0$ (as in the Spring-Mass model), the exponential term has a negative exponent. In this case (one real root), the system is **CRITICALLY DAMPED**.

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \alpha \pm i\beta$. Then the solution is:

$$y_h(t) = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

If $a, b, c > 0$, then $\alpha = -(b/2a) < 0$. In the case of complex roots, the system is said to be **UNDER-DAMPED**. If $\alpha = 0$ (this occurs when there is no damping), we get pure periodic motion, with period $2\pi/\beta$ or circular frequency β .

Solving the more general case

We had two methods for solving the more general equation:

$$y'' + p(t)y' + q(t)y = 0$$

but each method relied on already having one solution, $y_1(t)$. Given that situation, we can solve for y_2 (so that y_1, y_2 form a fundamental set), by one of two methods:

- By use of the Wronskian: There are two ways to compute this,

$$- W(y_1, y_2) = C e^{-\int p(t) dt} \quad (\text{This is from Abel's Theorem})$$

$$- W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = C e^{-\int p(t) dt}$

- Reduction of order, where $y_2 = v(t)y_1(t)$. Now substitute y_2 into the DE, and use the fact that y_1 solves the homogeneous equation, and the DE reduces to:

$$y_1 v'' + (2y_1' + p y_1) v' = 0$$

NOTE: I'd like for you to understand the technique- I'll give you the substitution if needed.

Finding the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters (but Variation of Parameters won't be on the exam).

Method of Undetermined Coefficients

This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, the table from the text:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1t + \dots + a_nt^n)$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t}(a_0 + a_1t + \dots + a_nt^n)$
$P_n(t)e^{\alpha t} \sin(\mu t)$ or $\cos(\mu t)$	$t^s e^{\alpha t} ((a_0 + a_1t + \dots + a_nt^n) \sin(\mu t) + (b_0 + b_1t + \dots + b_nt^n) \cos(\mu t))$

The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

Analysis of the Oscillator Model (3.7-3.8)

Given

$$mu'' + \gamma u' + ku = F(t)$$

we should be able to determine the constants from a given setup for a spring-mass system. Once that's done, be able to analyze the spring-mass system in some particular cases:

1. Unforced (The homogeneous equation, $F(t) = 0$)
 - (a) No damping: Natural frequency is $\sqrt{k/m}$
 - (b) With damping: Underdamped, Critically Damped, Overdamped
2. Periodic Forcing¹
 - (a) With no damping: Determine when Beating and Resonance occur.

$$u'' + \omega^2 u = F \cos(\omega_0 t)$$

“Beating” occurs when ω is close to ω_0 .

The circular frequency for one beat is $|\omega_0 - \omega|$. The amplitude of one beat: $2F/(\omega_0^2 - \omega^2)$.

“Resonance” occurs when $\omega = \omega_0$. Resonance forces the solution to become unbounded (can be very bad in the physical world!)

- (b) (This is Section 3.8, and is included here for completeness, but it won't be on Exam 2).

With damping: We changed the model to make our computations a bit easier:

$$u'' + pu' + qu = \cos(\omega t)$$

Then with $y_p = Ae^{i\omega t}$, we found that

$$A = \frac{1}{(q - \omega^2) + ip\omega} = \frac{1}{\alpha + \beta i}$$

¹The more general case of forcing we would use the Method of Undetermined Coefficients to solve.

Given this, we found that the amplitude R and phase angle δ of the forced response (also known as the particular part of the solution) is given by:

$$R = \frac{1}{|\alpha + \beta i|} \quad \delta = \tan^{-1}(\beta/\alpha)$$

Given this, be able to determine q or ω that will maximize the amplitude R (by differentiating, setting to zero.) Using that value of ω in the physical world can result in Resonance (blowing up the wine glass or a bridge!)