Solutions to the Complex Number Exercises

1. For each polynomial below, first find the roots by completing the square, then write the roots in polar form:

   (a) $x^2 - 2x + 10$
   SOLUTION: $x^2 - 2x + 10 = (x^2 - 2x + 1) + 9 = (x-1)^2 + 9 = 0$, so $x = 1 \pm 3i$.
   In polar form, $r = \sqrt{1^2 + 3^2} = \sqrt{10}$ and $\theta = \tan^{-1}(3)$. Therefore,

   $$1 \pm 3i = \sqrt{10}\,\mathrm{e}^{\pm \tan^{-1}(3)}$$

   (b) $x^2 + 4x + 5$
   SOLUTION: $x^2 + 4x + 5 = (x^2 + 4x + 4) + 1 = (x+2)^2 + 1 = 0$, so $x = -2 \pm i$. In polar form, for $-2 + i$, we have:

   $$r = \sqrt{4+1} = \sqrt{5} \qquad \theta = \tan^{-1}(-1/2) + \pi$$

   so we could write:

   $$-2 \pm i = \sqrt{5}\,\mathrm{e}^{\pm\theta}$$

   (c) $x^2 - x + 1$
   SOLUTION: $x^2 - x + 1 = \left(x^2 - x + \frac{1}{4}\right) + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = 0$ so $x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i$.
   In this case, we can find θ exactly since we have a "$1 - 2 - \sqrt{3}$" triangle.

   $$\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i = \mathrm{e}^{\pm\pi/3}$$

2. Show that:

   $$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

   SOLUTION: If $z = a + bi$, then $\bar{z} = a - bi$, so:

   $$z + \bar{z} = a + ib + (a - ib) = 2a = 2\mathrm{Re}(z)$$

   And similarly,

   $$z - \bar{z} = a + ib - (a - ib) = 2ib \quad \Rightarrow \quad \mathrm{Im}(z) = b = \frac{z - \bar{z}}{2i}$$

3. For the following, let $z_1 = -3 + 2i$, $z_2 = -4i$

   (a) Compute $z_1\bar{z}_2$, z_2/z_1
   SOLUTION:

   $$z_1\bar{z}_2 = (-3+2i)(4i) = (8i^2) - 12i = -8 - 12i$$
   $$\frac{z_2}{z_1} = \frac{-4i}{-3+2i} \cdot \frac{-3-2i}{-3-2i} = \frac{-8+12i}{3^2+4^2} = -\frac{8}{13} + \frac{12}{13}i$$

   (b) Write z_1 and z_2 in polar form.
   SOLUTION: For z_1, the point $(-3, 2)$ is in Quadrant II. To find the argument (angle) for z_1, we have to add π to the arctangent:

   $$\theta = \tan^{-1}\left(\frac{2}{-3}\right) + \pi \approx 2.554 \text{ rad}$$

   The length is: $r = \sqrt{3^2 + 2^2} = \sqrt{13}$.

We can verify this, by checking that:

$$\sqrt{13}\cos(\theta) \approx -3 \qquad \sqrt{13}\sin(\theta) \approx 2$$

Writing the answer in polar form, $re^{i\theta}$,

$$-3 + 2i = \sqrt{13}\mathrm{e}^{2.554i}$$

For z_2, it is much simpler: $r = 4$ and $\theta = -\pi/2$. Therefore,

$$-4i = r\mathrm{e}^{i\theta} = 4\mathrm{e}^{-i\pi/2}$$

4. In each problem, rewrite each of the following in the form $a + bi$:

   (a) e^{1+2i}

   $$\mathrm{e}\left(\cos(2) + i\sin(2)\right)$$

   (b) e^{2-3i}

   $$\mathrm{e}^2\left(\cos(-3) + i\sin(-3)\right) = \mathrm{e}^2\left(\cos(3) - i\sin(3)\right)$$

   (c) $\mathrm{e}^{i\pi}$

   $$\cos(\pi) + i\sin(\pi) = -1$$

   (d) 2^{1-i} First, notice that $2^{1-i} = 2 \cdot 2^{-i}$, so we'll compute 2^{-i} below:

   $$2^{-i} = \mathrm{e}^{\ln(2^{-i})} = \mathrm{e}^{-i\ln(2)} = \cos(-\ln(2)) + i\sin(-\ln(2)) = \cos(\ln(2)) - i\sin(\ln(2))$$

   Therefore, 2^{1-i} can be expressed as:

   $$2(\cos(\ln(2)) - i\sin(\ln(2)))$$

   (e) $\mathrm{e}^{2-\frac{\pi}{2}i}$

   $$\mathrm{e}^2\left(\cos(-\pi/2) + i\sin(-\pi/2)\right) = -i\mathrm{e}^2$$

   (f) π^i

   $$\pi^i = \mathrm{e}^{\ln(\pi^i)} = \mathrm{e}^{i\ln(\pi)} = \cos(\ln(\pi)) + i\sin(\ln(\pi))$$

5. For fun, compute the logarithm of each number:

   (a) $\ln(-3) = \ln(3\mathrm{e}^{i\pi/2}) = \ln(3) + \frac{\pi}{2}\,i$

   (b) $\ln(-1+i) = \ln(\sqrt{2}\mathrm{e}^{3\pi/4\,i}) = \ln(\sqrt{2}) + \frac{3\pi}{4}\,i$

   (c) $\ln(2\mathrm{e}^{3i})$