Exam 2 Summary

The exam will cover material from Section 3.1 to 3.8 except for 3.6 (Variation of Parameters). Here is a summary of that information.

Existence and Uniqueness:

Given the second order linear IVP,

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = v_0$$

If there is an open interval I on which p, q, and g are continuous an contain t_0, then there exists a unique solution to the IVP, valid on I (and may contain the endpoints of I, if the functions are also continuous there).

Structure and Theory (Mostly 3.2)

The goal of the theory was to establish the structure of solutions to the second order IVP:

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = v_0$$

In summary, the general solution is a sum of the homogeneous and particular solutions. The details of that are given in the theorems below (what do you need to form the full homogeneous solution, for example):

1. Vocabulary: Linear operator, general solution, fundamental set of solutions, linear combination.
2. **Solving the homogeneous equation, $L(y) = 0$.**

   If $y_1, y_2, \ldots, y_k$ each solves the homogeneous equation with linear operator L, then so does any linear combination,

   $$c_1y_1 + \cdots c_ky_k$$

   That is,

   $$L(c_1y_1 + \cdots c_ky_k) = c_1L(y_1) + \cdots c_kL(y_k) = 0$$

   and this is the **superposition principle**.

   Once we find a bunch of candidate solutions to $L(y) = 0$, which functions will be sufficient to write the solution using arbitrary intial conditions? That's the purpose of the Wronskian:

3. **The Wronskian and the fundamental set**

   For the second order linear homogeneous equation, $y'' + p(t)y' + q(t)y = 0$, if we can find two solutions, y_1, y_2 so that

   $$W(y_1, y_2)(t_0) \neq 0$$

   then y_1, y_2 forms a fundamental set of solutions. That is, the general solution to the homogeneous equation can be written as

   $$y_h(t) = C_1y_1(t) + C_2y_2(t)$$

   Abel's theorem generalizes this a bit- Rather than the Wronskian only at a single point t_0, is says more:

4. **Abel's theorem**.

   If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian, $W(y_1, y_2)$, is either always zero or never zero on the interval for which the solutions are valid.

   That is because the Wronskian may be computed as:

   $$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\,dt}$$

   Comment: Abel's theorem gives us a new way of computing the Wronskian (the first being by using the definition).

5. The last piece of theory is just in showing that the full solution to the non-homogeneous differential equation is given by:

   $$y(t) = y_h(t) + y_p(t)$$

   where $y_h(t)$ is the homogeneous solution, and $y_p(t)$ is the particular solution. This is actually easy to show, since $y'' + p(t)y' + q(t)y = g(t)$ can be written as $L(y) = g(t)$, where L is a linear operator. Then

   $$L(y_h + y_p) = L(y_h) + L(y_p) = 0 + g(t) = g(t)$$

In summary, the theory tells us that y_h requires two functions for the fundamental set, and then we needed a way of getting one particular solution.

Solve $ay'' + by' + cy = 0$ for $y_h(t)$.

To solve $ay'' + by' + cy = 0$ we use the **ansatz** $y = \mathrm{e}^{rt}$. Then we form the associated **characteristic equation**:

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way:

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2. The general solution is:

  $$y_h(t) = c_1\mathrm{e}^{r_1 t} + c_2\mathrm{e}^{r_2 t}$$

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

  $$y_h(t) = \mathrm{e}^{-(b/2a)t}\,(C_1 + C_2 t)$$

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \alpha \pm i\beta$. Then the solution is:

  $$y_h(t) = \mathrm{e}^{\alpha t}\,(C_1\cos(\beta t) + C_2\sin(\beta t))$$

Solving $y'' + p(t)y' + q(t)y = 0$ for $y_h(t)$

We had two methods for solving the more general equation:

$$y'' + p(t)y' + q(t)y = 0$$

but each method relied on already having one solution, $y_1(t)$. Given that situation, we can solve for y_2 (so that y_1, y_2 form a fundamental set), by one of two methods:

- By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$ (This is from Abel's Theorem)

– $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

These should be equal, and y_2 is the unknown in the first order ODE: $y_1 y_2' - y_2 y_1' = C\mathrm{e}^{-\int p(t)\,dt}$

- Reduction of order, where $y_2 = v(t) y_1(t)$. Now substitute y_2 into the DE, and use the fact that y_1 solves the homogeneous equation, and the DE reduces to:

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

NOTE: I'd like for you to understand the technique- I'll give you the substitution if needed- you don't need to have this technique memorized for the exam.

Finding the particular solution, $y_p(t)$.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters (but Variation of Parameters won't be on the exam).

Method of Undetermined Coefficients

This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, the table from the text:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1 t + \ldots a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_0 + a_1 t + \ldots + a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s \mathrm{e}^{\alpha t}\left((a_0 + a_1 t + \ldots + a_n t^n)\sin(\mu t)\right.$ $\left.+ (b_0 + b_1 t + \ldots + b_n t^n)\cos(\mu t)\right)$

The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

Building and Analyzing Solutions to the Spring Mass Model (3.7-3.8)

Given

$$mu'' + \gamma u' + ku = F(t)$$

we should be able to determine the constants from a given setup for a spring-mass system. Once that's done, be able to analyze the spring-mass system in some particular cases:

1. No Forcing (so $mu'' + \gamma u' + ku = 0$)
   (a) No damping, so $mu'' + ku = 0$: Natural frequency is $\sqrt{k/m}$ (be able to write the solution to the ODE).
   (b) With damping, so $mu'' + \gamma u' + ku = 0$.
   Be able to solve the system, and state if the system is Underdamped, Critically Damped, Overdamped.
2. Periodic Forcing
   (a) With no damping: $u'' + \omega^2 u = F\cos(\omega t)$
      - "Beating" occurs when ω is close to ω_0, and in that case, the circular frequency for one beat is $|\omega_0 - \omega|$.
      - "Resonance" occurs when $\omega = \omega_0$. Resonance forces the solution to become unbounded (can be very bad in the physical world!)

- We should be able to solve the ODE, but there isn't anything special here- Just use the method of undetermined coefficients.

(b) With damping: $u'' + pu' + qu = F\cos(\omega t)$

To solve this equation, we use the standard method (undetermined coefficients). However, to analyze the solutions, we rewrote the particular solution as $R\cos(\omega t - \delta)$, and we found that the value of R was given by:

$$R = \frac{1}{\sqrt{(q-\omega^2)^2 + p^2\omega^2}}$$

This expression would be given to you if needed on the exam. The importance here is that the amplitude is a function of the quantities p, q and ω. Therefore, for example, given that p, q are fixed, we can determine the value of ω that maximizes the amplitude R. This would represent the **resonant frequency** (in the case that damping is present).

The maximum is found using the standard technique of setting the derivative of R equal to zero, etc. Note that we could optimize R using p or q instead of ω.

Be sure to look over the page of homework questions that is a substitute for Section 3.8 to get an idea of the kinds of questions for this section.