Linear Operators and Cramer's Rule

1. Let $R(f)$ be the operator defined by: $R(f) = f''(t) + 3t^2 f(t)$.
   For example, $R(\mathrm{e}^{2t} + 5t) = 4\mathrm{e}^{2t} + 3t^2\mathrm{e}^{2t} + 15t^3$. Find $R(f)$ for each f below:

   (a) $f(t) = t^2$ (b) $f(t) = \sin(3t)$ (c) $f(t) = 2t - 5$

2. Let R be the operator defined in the previous problem. Show that R is a linear operator. Hint: Show that $R(cf + dg) = cR(f) + dR(g)$ for functions f, g and scalars c, d.

3. Let $F(y) = y'' + y - 5$. Explain why F is not linear.

4. Find the operator associated with the given differential equation, and classify it as linear or not linear:

   (a) $y' = ty^2 + \cos(t)$
   (b) $y'' = 4y' + 3y + \sin(t)$
   (c) $y' = \mathrm{e}^t y + 5$
   (d) $y'' = -\cos(y) + \cos(t)$

5. Use Cramer's Rule to solve the following systems:

   (a) $\begin{aligned} C_1 + C_2 &= 2 \\ -2C_1 - 3C_2 &= 3 \end{aligned}$

   (b) $\begin{aligned} C_1 + C_2 &= y_0 \\ r_1C_1 + r_2C_2 &= v_0 \end{aligned}$

   (c) $\begin{aligned} C_1 + C_2 &= 2 \\ 3C_1 + C2 &= 1 \end{aligned}$

   (d) $\begin{aligned} 2C_1 - 5C_2 &= 3 \\ 6C1 - 15C_2 &= 10 \end{aligned}$

   (e) $\begin{aligned} 2x - 3y &= 1 \\ 3x - 2y &= 1 \end{aligned}$

6. Suppose L is a linear operator. Let y_1, y_2 each solve the equation $L(y) = 0$ (so that $L(y_1) = 0$ and $L(y_2) = 0$). Show that anything of the form $c_1y_1 + c_2y_2$ will also solve $L(y) = 0$.

   This means that if we have two solutions to the homogeneous linear equation, then any combination of the form $c_1y_1 + c_2y_2$ is also a solution to the homogeneous equation.