Sample Questions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$
   SOLUTION: False. The characteristic equation is for the homogeneous equation, $r^2 + r + 1 = 0$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$
   SOLUTION: False. The characteristic equation was defined only for DEs with constant coefficients, since our ansatz depended on constant coefficients.

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.
   SOLUTION: True. For a linear operator L, it is always true that $L(0) = 0$.

   (d) In using the Method of Undetermined Coefficients, the ansatz
   $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   SOLUTION: False- We have to be able to choose the coefficients for each polynomial (for the sine and cosine) independently of each other. In the form:

   $$(Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$$

   the polynomials for the sine and cosine are constant multiples of each other, which may not necessarily hold true. That's why we need one polynomial for the sine, and one for the cosine (so the second guess is the one to use).

   (e) The operator $T(y) = y' + t^2 y + 1$ is a linear operator (in y).
   SOLUTION: False- The "+1" term makes the operator not linear. For example, $T(cy) = cy' + ct^2 y + 1$, but $c(T(y)) = cy' + ct^2 y + c$.

2. (a) First, solve the DE: $y'' + 4y' + 3y = 0$. (b) Use Cramer's rule to find the constants, if the initial conditions are $y(0) = 1$, $y'(0) = \alpha$.

   SOLUTION: The solutions to the characteristic equation are $r = -1, -3$, so that the general solution is

   $$y(t) = C_1 e^{-t} + C_2 e^{-3t}$$

   Applying the initial conditions gives:

   $$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 - 3C_2 &= \alpha \end{aligned} \quad \Rightarrow \quad C_1 = \frac{\begin{vmatrix} 1 & 1 \\ \alpha & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix}} = \frac{-3-\alpha}{-2}, C_2 = \frac{\begin{vmatrix} 1 & 1 \\ -1 & \alpha \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix}} = \frac{\alpha+1}{-2}$$

3. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

   SOLUTION: The operator is found by moving the terms with y to the left side, and everything else to the right. What remains on the left is the operator. In this case, $T(y) = y' - y^2$.

   Because we're squaring y, we expect it NOT to be linear, but we can check:

   $$T(y_1 + y_2) = (y_1 + y_2)' - (y_1 + y_2)^2 = y_1' - y_1^2 + y_2' - y_2^2 - 2y_1 y_2$$

   The extra term on the end makes this expression not equal to $T(y_1) + T(y_2)$. Similarly, we would check if $T(cy) = cT(y)$:

   $$T(cy) = cy' - (cy)^2 = cy' - c^2 y^2 \neq c(y' - y^2)$$

   Therefore, the second part of linearity does not hold either.

4. What do we need to check in order to see if two functions, $y_1.y_2$ form a **fundamental set of solutions** to a given second order linear homogeneous DE?

   SOLUTION: The functions y_1, y_2 will form a fundamental set of solutions if (1) both satisfy the DE separately, and (2) they satisfy $W(y_1, y_2)(t_0) \neq 0$.

5. If $W(f, g) = t^2 \mathrm{e}^t$ and $f(t) = t$, find $g(t)$.

   SOLUTION:
$$W(f,g) = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t) \quad \Rightarrow \quad tg' - g = t^2\mathrm{e}^t$$

   Solving this, divide by t, compute the integrating factor, etc:
$$g' - \frac{1}{t}g = t\mathrm{e}^t \quad \Rightarrow \quad (g/t)' = \mathrm{e}^t \quad \Rightarrow \quad g/t = \mathrm{e}^t + C \quad \Rightarrow \quad g(t) = t\mathrm{e}^t + Ct$$

   (But notice that t is actually $f(t)$, so we don't need the Ct part).

6. If y_1, y_2 form a fundamental set of solutions to: $t^2y'' - 2y' + (3 + t)y = 0$, and if $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(4)$.

   SOLUTION: (This is exercise 35 in 3.2) Compute the Wronskian using Abel's Theorem:
$$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\, dt} = C\mathrm{e}^{-\int -2/t^2\, dt} = C\mathrm{e}^{-2/t}$$

   Now, if $W(y_1, y_2)(2) = 3$, then we can find C:
$$C \cdot \mathrm{e}^{-1} = 3 \quad \Rightarrow \quad C = 3e$$

   Now we can compute the Wronskian at $t = 4$:
$$W(y_1, y_2)(4) = 3e \cdot \mathrm{e}^{-2/4} = 3\sqrt{e}$$

7. Given that $y_1 = \frac{1}{t}$ solves the differential equation: $t^2y'' - 2y = 0$, find a fundamental set of solutions using Abel's Theorem. SOLUTION: First, rewrite the differential equation in standard form:
$$y'' - \frac{2}{t^2}y = 0$$

   Then $p(t) = 0$ and $W(y_1, y_2) = C\mathrm{e}^0 = C$. On the other hand, the Wronskian is:
$$W(y_1, y_2) = \frac{1}{t}y_2' + \frac{1}{t^2}y_2$$

   Put these together:
$$\frac{1}{t}y_2' + \frac{1}{t^2}y_2 = C \quad y_2' + \frac{1}{t}y_2 = Ct$$

   The integrating factor is t,
$$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1t^3 + C_2 \quad \Rightarrow \quad C_1t^2 + \frac{C_2}{t}$$

   Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

8. Give the general solution:

   (a) $y'' - 3y' - 10y = 0$

   SOLUTION: The characteristic equation is $r^2 - 3r - 10 = 0$. The factors of 10 are $2, 5$ and so those work: $(r - 5)(r + 2) = 0$, and $r = -2, 5$.
$$y(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{5t}$$

(b) $y'' + 4y' + 4y = 0$
SOLUTION: The characteristic equation is a perfect square:

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = 0 \quad r = -2, -2$$

Therefore,

$$y(t) = C_1 \mathrm{e}^{-2t} + C_2 t\mathrm{e}^{-2t} = \mathrm{e}^{-2t}(C_1 + C_2 t)$$

(c) $y'' - 4y' + 5y = 0$
SOLUTION: Complex roots: $r^2 - 4r + 5 = 0$ becomes $r^2 - 4r + 4 = -1$ by completing the square, or

$$(r-2)^2 = -1 \quad \Rightarrow \quad r = 2 \pm i$$

The solution is then:

$$y(t) = C_1 \mathrm{e}^{2t}\cos(t) + C_2 \mathrm{e}^{2t}\sin(t) = \mathrm{e}^{2t}(C_1\cos(t) + C_2\sin(t))$$

9. Suppose the roots to the characteristic equation are as given below. Write the general solution to the DE, and write down what the second order linear homogeneous DE was.

(a) $r = -2, 3$
SOLUTION: The characteristic equation becomes $(r+2)(r-3) = 0$, or $r^2 - r - 6 = 0$, and that would come from the DE: $y'' - y' - 6y = 0$. The solution to the DE is $y(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{3t}$.

(b) $r = 1, 1$
SOLUTION: The characteristic equation becomes $(r-1)^2 = 0$, or $r^2 - 2r + 1 = 0$, and that would come from the DE: $y'' - 2y' + y = 0$. The solution to the DE is $y(t) = \mathrm{e}^t(C_1 + C_2 t)$.

(c) $r = 2 \pm 3i$
SOLUTION: We want $(r-2)^2 = -9$, so that when we take the square root, we get the desired result. Simplifying, we get $r^2 - 4r + 4 = -9$, which "simplifies" to $r^2 - 4r + 13 = 0$. That would come from the DE: $y'' - 4y' + 13y = 0$. The solution to the DE is $y(t) = \mathrm{e}^{2t}(C_1\cos(3t) + C_2\sin(3t))$.

10. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $\mathrm{e}^{(3-2i)t}$ (iii) $\mathrm{e}^{i\pi}$

NOTE for the SOLUTION: Remember that for any non-negative number A, we can write $A = \mathrm{e}^{\ln(A)}$.

- $2^{i-1} = \mathrm{e}^{ln(2^{i-1})} = \mathrm{e}^{(i-1)\ln(2)} = \mathrm{e}^{-\ln(2)}\mathrm{e}^{i\ln(2)} = \frac{1}{2}\left(\cos(\ln(2)) + i\sin(\ln(2))\right)$
- $\mathrm{e}^{(3-2i)t} = \mathrm{e}^{3t}\mathrm{e}^{-2ti} = \mathrm{e}^{3t}\left(\cos(-2t) + i\sin(-2t)\right) = \mathrm{e}^{3t}\left(\cos(2t) - i\sin(2t)\right)$
  (Recall that cosine is an even function, sine is an odd function).
- $\mathrm{e}^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

11. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

SOLUTIONS:

(i) $r = \sqrt{1+3} = 2$, $\theta = -2\pi/3$ (look at its graph, use 30-60-90 triangle):

$$-1 - \sqrt{3}i = 2\mathrm{e}^{-\frac{2\pi}{3}i}$$

(ii) $3i = 3\mathrm{e}^{\frac{\pi}{2}i}$

(iii) $-4 = 4\mathrm{e}^{\pi i}$

(iv) $\sqrt{3} - i = 2\mathrm{e}^{-\frac{\pi}{6}i}$

12. Write each expression as $R\cos(\omega t - \delta)$

(a) $3\cos(2t) + 4\sin(2t)$

SOLUTION: For each of these, think of $A\cos(2t)+B\sin(2t)$ as defining a complex number $A+iB$. Then R is the magnitude and δ is the angle for $A+iB$. In this particular case, we see that (A, B) is in Quadrant I, so δ does not need an extra π added to it:

$$R = \sqrt{9+16} = 5 \qquad \delta = \tan^{-1}(4/3) \Rightarrow 3\cos(2t) + 4\sin(2t) = 5\cos(2t - \tan^{-1}(4/3))$$

(b) $-\cos(t) + \sqrt{3}\sin(t)$

SOLUTION: Note that in this case, $(-1, \sqrt{3})$ is in Quadrant II, so add π to δ. Also, notice that the angle δ is coming from a triangle with side $1, 2, \sqrt{3}$ (or 30-60-90). In this case,

$$R = \sqrt{1+3} = 2 \qquad \delta = \tan^{-1}(-\sqrt{3}) = -\pi/3, \quad \Rightarrow \quad -\cos(t) + \sqrt{3}\sin(t) = 2\cos(t - (2\pi/3))$$

(c) $4\cos(3t) - 2\sin(3t)$

SOLUTION: In this case, $(4, -2)$ is coming from Quadrant IV, so no need to add π to δ. We don't have a special triangle in this case.

$$R = \sqrt{16+4} = 2\sqrt{5} \qquad \delta = \tan^{-1}(-1/2) \quad \Rightarrow \quad 4\cos(3t) - 2\sin(3t) = 2\sqrt{5}\cos(t - \tan^{-1}(-1/2))$$

13. Practice setting up the Spring-Mass model: If you need it, $g \approx 9.8 = \frac{49}{5}$.

(a) Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

SOLUTION: We can find the differential equation:

$$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + 100ku = 0$$

Then the system is *critically damped* if the discriminant (from the quadratic formula) is zero:

$$b^2 - 4ac = 25 - 4 \cdot 100k = 0 \quad \Rightarrow \quad k = \frac{25}{400} = \frac{1}{16}$$

The system is *underdamped* if the discriminant is negative:

$$25 - 400k < 0 \quad \Rightarrow \quad k > \frac{1}{16}$$

(NOTE: I'll try to make sure the numbers work out nicely on the exam).

(b) A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

SOLUTION:

We use Hooke's Law at equilibrium: $mg - kL = 0$, or

$$k = \frac{mg}{L} = \frac{(1/2)(9.8)}{(1/20)} = (10)(9.8) = 98$$

For the second part, a stiff spring will not stretch, so L will be small (and k would therefore be large), and a spring that is not stiff will stretch a great deal (so that k will be smaller).

(c) It takes 6 N of force to stretch a certain spring 3 meters. A mass of $\frac{1}{2}$ kg is attached to a spring. (a) Find the spring constant, and (b) if the damping constant is 2, write the differential equation for the motion of the mass.

SOLUTION: For the spring constant, $6 = kL = 3k$, so $k = 2$. If $\gamma = 2$ as well, then the DE for the motion of the mass is

$$\frac{1}{2}u'' + 2u' + 2u = 0$$

(d) Given the model of motion of the mass on a spring is given by

$$\frac{1}{2}u'' + \gamma u' + 8u = 0$$

Find γ so that the spring is *underdamped*, *critically damped*, *overdamped*.

SOLUTION: We're considering the discriminant of the characteristic equation, which in this case is given by

$$\gamma^2 - 4mk = \gamma^2 - 4(\frac{1}{2})(8) = \gamma^2 - 16$$

In each case, we may assume $\gamma > 0$ since it represents damping in a physical system.

- For $\gamma^2 - 16 > 0$, we must have $\gamma > 4$. In this case, the system is overdamped.
- If $\gamma = 4$, the system is critically damped.
- $0 < \gamma < 4$, the system is underdamped.

14. Solve. If there are initial conditions, solve for all constants, otherwise, find the general solution.

(a) $u'' + u = 3t + 4$, $u(0) = 0, u'(0) = 0$.

The solution is: $u(t) = -4\cos(t) - 3\sin(t) + 3t + 4$.

Details: The characteristic equation is $r^2 + 1 = 0$, so the roots are $r = \pm i$, and $u_h(t) = C_1\cos(t) + C_2\sin(t)$.

For the particular solution, guess $u_p(t) = At + B$. Substituting this into the DE, we get

$$0 + (At + B) = 3t + 4$$

So in this case, the solution is very easy: $u_p(t) = 3t + 4$. Putting the solutions together,

$$u(t) = C_1\cos(t) + C_2\sin(t) + 3t + 4$$

Now put in the initial conditions $u(0) = 0$ and $u'(0) = 0$:

$$\begin{aligned} 0 &= C_1 + 4 \\ 0 &= C_2 + 3 \end{aligned} \quad \Rightarrow \quad C_1 = -4, C_2 = -3$$

(b) $u'' + u = \cos(2t)$, $u(0) = 0$, $u'(0) = 0$

In this case, the homogeneous part didn't change from the previous problem, and the guess for the particular part is

$$u_p = A\cos(2t) + B\sin(2t) \quad u_p' = 2B\cos(2t) - 2A\sin(2t) \quad u_p'' = -4A\cos(2t) - 4B\sin(2t)$$

Therefore, we get

$$(-4A\cos(2t) - 4B\sin(2t)) + (A\cos(2t) + B\sin(2t)) = \cos(2t) \quad \Rightarrow -3A = 1, B = 0$$

The solution thus far is

$$u(t) = C_1\cos(t) + C_2\sin(t) - \frac{1}{3}\cos(2t)$$

Using the initial conditions: $0 = C_1 - \frac{1}{3}$ and $0 = C_2$. Now we can write:

$$u(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t)$$

(c) $u'' + u = \cos(t)$, $u(0) = 0$, $u'(0) = 0$ (And please compare to the previous problem).

SOLUTION: The solution gets a little messy because we now have to multiply our guess by t, but we can differentiate and substitute as usual. We'll group terms with $\cos(t)$ and $\sin(t)$ together:

$$u_p(t) = t(A\cos(t) + B\sin(t)), \quad u_p' = (At + B)\cos(t) + (-Bt + A)\sin(t)$$

In the second derivative, we get $u_p'' = (-At + 2B)\cos(t) - (Bt + 2A)\sin(t)$.

Putting these together in the DE:

$$2B\cos(t) + 2A\sin(t) = \cos(t) \quad \Rightarrow \quad B = 1/2, A = 0$$

Now we get

$$u(t) = C_1\cos(t) + C_2\sin(t) + \frac{t}{2}\sin(t)$$

With zero initial conditions, $u(t) = \frac{t}{2}\sin(t)$.

For the comparison: This function blows up because of resonance- The forcing frequency matches the natural frequency. In the previous problem, these were unequal.

(This problem had more algebra than you'll typically see on the exam).

(d) $y'' + 4y' + 4y = \mathrm{e}^{-2t}$

SOLUTION: Find the homogeneous part of the solution first, so the characteristic equation is $r^2 + 4r + 4 = 0$, or $(r + 2)^2 = 0$, or $r = -2, -2$. Therefore, $y_h(t) = C_1\mathrm{e}^{-2t} + t\mathrm{e}^{-2t}$. For the particular solution, we guess $y_p(t) = \mathrm{e}^{-2t}$, but we'll have to multiply it by t^2.

Solving for the coefficient:

$$y_p = At^2\mathrm{e}^{-2t}, \quad y_p' = 2At\mathrm{e}^{-2t} - 2At^2\mathrm{e}^{-2t}, \quad y_p'' = 2A\mathrm{e}^{-2t} - 8At\mathrm{e}^{-2t} + 4At^2\mathrm{e}^{-2t}$$

Substitute into the DE (factor out the $A\mathrm{e}^{-2t}$ term)

$$A\mathrm{e}^{-2t}((2 - 8t + 4t^2) + 4(2t - 2t^2) + 4t^2) = \mathrm{e}^{-2t} \quad \Rightarrow \quad 2A = 1$$

Therefore, the full solution is

$$y(t) = \mathrm{e}^{-2t}(C_1 + C_2 t) + \frac{1}{2}t^2\mathrm{e}^{-2t}$$

(e) $y'' - 2y' + y = t\mathrm{e}^t + 4$, $y(0) = 1$, $y'(0) = 1$.

Breaking apart our solution:

- $y_h = \mathrm{e}^t(C_1 + C_2 t)$
- For the constant forcing, $y_{p_1} = A$. It's easy to see that $y_{p_1}(t) = 4$.
- For the other forcing function, we need to multiply our guess by t^2. The algebra does get a bit messy, but this is very similar to the last problem:

$$y_p(t) = t^2(At + B)\mathrm{e}^t$$

  We should find that $A = 1/6, B = 0$, so $y_p(t) = \frac{1}{6}t^3\mathrm{e}^t$

Putting it together, we can find C_1, C_2. We should find that the full solution is given by

$$y(t) = \mathrm{e}^t(-3 + 4t) + \frac{1}{6}t^3\mathrm{e}^t + 4$$

(Sorry for the heavy algebra- Typically there won't be that much on an exam).

15. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}t^2 - t$$

SOLUTION: We can see that the roots to the characteristic equation are $r = 0$ and $r = -1$, so that the characteristic equation could be expressed as $r(r+1) = 0$, or $r^2 + r = 0$. This gives the homogeneous part of the DE as $y'' + y' = 0$.

If $y_p(t) = \frac{1}{2}t^2 - t$ is a solution, then we can substitute it into the DE as see what we get:

$$y_p'' + y_p' = (1) + (t-1) = t \quad \Rightarrow \quad y'' + y' = t$$

This is our full differential equation.

16. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

SOLUTION: Write in standard form first-

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

17. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

SOLUTION: This purpose of this question is to see if we can use the properties of linearity to get at the answer.

We see that: $L(3e^{2t}) = -9e^{2t}$, or $L(e^{2t}) = -3e^{2t}$ so:

$$L\left(-\frac{1}{3}e^{2t}\right) = e^{2t}$$

And for the second part,

$$L(t^2 + 3t) = 5t^2 + 3t - 16 \quad \Rightarrow \quad L((-2)(t^2 + 3t)) = -10t^2 + 6t - 32$$

The particular solution is therefore:

$$y_p(t) = -2(t^2 + 3t) - \frac{1}{3}e^{2t}$$

since we have shown that

$$L\left(-2(t^2 + 3t) - \frac{1}{3}e^{2t}\right) = -10t^2 + 6t - 32 + e^{2t}$$

18. For each DE below, use the Method of Undetermined Coefficients to give the final form of your guess for the particular solution, $y_p(t)$. Do NOT solve for the coefficients.

(a) $y'' + 3y' = t^3 + t^2 \mathrm{e}^{-t} + \sin(3t)$

SOLUTION: The idea is that we'll split our solution into several parts- One for the homogeneous equation, one for the polynomial, one for the middle exponential term and one for the trig function. The final guess for the particular solution will be the sum of the three particular guesses.

- For y_h, we see $r = 0$ and $r = -3$, so $y_h(t) = C_1 + C_2 \mathrm{e}^{-3t}$.
- For t^3, we first guess $y_p = At^3 + Bt^2 + Ct + D$. However, the constant D is part of the homogeneous solution $(C_1 \cdot 1)$, so multiply by t:

$$y_{p_1}(t) = t(A_1 t^3 + B_1 t^2 + C_3 t + D_1)$$

- For the degree 2 poly times the exponential, we're all clear:

$$y_{p_2}(t) = (A_2 t^2 + B_2 t + C_4)\mathrm{e}^{-t}$$

- For the sine term, we're also clear of y_h:

$$y_{p_3}(t) = A_3 \cos(3t) + B_3 \sin(3t)$$

So again, the full guess is $y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$

(b) $y'' + y = t(1 + \sin(t))$

SOLUTION: Again, we break our solution up into pieces. First the homogeneous part, then we'll deal with "t", then "tsin(t)".

- For y_h, $r = \pm i$, so $y_h = C_1 \cos(t) + C_2 \sin(t)$.
- For the t term: $y_{p_1}(t) = At + B$, and we don't need to multiply the guess.
- For the $t\sin(t)$ term, we need a full poly of degree 1 for both the sine and cosine terms:

$$y_{p_2}(t) = (A_2 t + B_2)\cos(t) + (D_2 t + E_2)\sin(t)$$

And because $\cos(t), \sin(t)$ are in the homogeneous part, we need to multiply by t, and this gives us the final form for $y_{p_2}(t)$:

$$y_{p_2}(t) = t((A_2 t + B_2)\cos(t) + (D_2 t + E_2)\sin(t))$$

And the full guess is $y_p(t) = y_{p_1}(t) + y_{p_2}(t)$.

(c) $y'' - 5y' + 6y = \mathrm{e}^{2t}(3t + 4)\sin(t)$

SOLUTION: Same technique as before- Break up the solution into the homogeneous part and the particular part.

- y_h : The char eqn is $(r-2)(r-3) = 0$, so $r = 2, 3$ and the homogeneous part of the solution is $C_1 \mathrm{e}^{2t} + C_2 \mathrm{e}^{3t}$.
- For y_p, think of the given expression as an exponential times a poly of deg 1, times the sine. We need a separate poly of degree 1 for the sine and cosine in our guess:

$$y_p(t) = \mathrm{e}^{2t}(At + B)\cos(t) + \mathrm{e}^{2t}(Dt + E)\sin(t)$$

And we don't need to multiply by t in this case.

(d) $y'' + 2y' + 2y = 3\mathrm{e}^{-t} + 2\mathrm{e}^{-t}\cos(t) + 4\mathrm{e}^{-t}t^2\sin(t)$

SOLUTION: As usual, break up the solution into three parts counting the homogeneous part - Note that we can deal with both the sine and cosine term in one guess.

- For y_h, we see that $r = -1, -1$, so y_h is using e^{-t} and $t\mathrm{e}^{-t}$ for the fundamental set.
- For the $3\mathrm{e}^{-t}$ term, we guess $y_{p_1}(t) = A\mathrm{e}^{-t}$, but we have to multiply that by t^2, so our final form is

$$y_{p_1}(t) = At^2 \mathrm{e}^{-t}$$

- For the other two terms, we're clear of the homogeneous solution so we won't multiply by t. We see a polynomial of degree 2 being used for the sine term, so we'll use a full poly of deg 2 for both the cosine and sine in our guess. Notice that automatically includes the form for the cosine part.

$$y_{p_2}(t) = \mathrm{e}^{-t}((A_1t^2 + A_2t + A_3)\cos(t) + (B_1t^2 + B_2t + B_3)\sin(t))$$

Now, just like before, $y_p(t) = y_{p_1}(t) + y_{p_2}(t)$.

19. Each equation below exhibits either **beating**, **resonance**, or **neither**. Label each one with B, R, or N.

SOLUTION: Recall that the form we're looking at is $y'' + \omega_0^2 y = \cos(\omega t)$ (or $\sin(\omega t)$). If $\omega_0 \approx \omega$, we get beating. If $\omega_0 = \omega$, we get resonance. Otherwise, label as "neither".

(a) $y'' + 3y = \cos(3t)$ N

(b) $y'' + 9y = \cos(3t)$ RESONANCE

(c) $y'' + (1.1)^2 y = \sin(t)$ BEATING

(d) $y'' + 3y = \cos(9t)$ N