Exercise Set 3 (HW to replace 7.3, 7.5)

This homework is all about solving for eigenvalues and eigenvectors, and we'll also do some visualization and classification of equilibria.

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

   SOLUTION: It's probably best to break it apart-

$$\begin{array}{ll} x_1(t) = C_1 e^{-t} + 2C_2 e^{2t} & x_1'(t) = -C_1 e^{-t} + 4C_2 e^{2t} \\ x_2(t) = 2C_1 e^{-t} + C_2 e^{2t} & \Rightarrow \qquad x_2'(t) = -2C_1 e^{-t} + 2C_2 e^{2t} \end{array}$$

   Verify that these expressions for x_1', x_2' are indeed found by

$$x_1' = 3x_1 - 2x_2 \quad \text{and} \quad x_2' = 2x_1 - 2x_2$$

   For example, the first expression below should be x_1' and the second should be x_2'

$$\begin{array}{ll} 3x_1(t) \quad = 3C_1 e^{-t} + 6C_2 e^{2t} & 2x_1(t) \quad = 2C_1 e^{-t} + 4C_2 e^{2t} \\ \underline{-2x_2(t) \quad = -4C_1 e^{-t} - 2C_2 e^{2t}} & \underline{-2x_2(t) \quad = -4C_1 e^{-t} - 2C_2 e^{2t}} \\ \qquad\qquad = -C_1 e^{-t} + 4C_2 e^{2t} & \qquad\qquad = -2C_1 e^{-t} + 2C_2 e^{2t} \end{array}$$

   And those do check out.

2. For each matrix, find the eigenvalues and eigenvectors. Note that they may be complex (when solving the quadratic equation).

   (a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 4, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$
       SOLUTION: $\lambda^2 - 2\lambda + 5 = 0$, so $\lambda^2 - 2\lambda + 1 = -4$ and $\lambda = 1 \pm 2i$.
       For $\lambda = 1 + 2i$, we have:

$$(3 - (1 + 2i))v_1 - 2v_2 = 0 \quad \Rightarrow \quad (2 - 2i)v_1 - 2v_2 = 0 \quad \Rightarrow \quad (1 - i)v_1 - v_2 = 0$$

   so we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$. For $\lambda = 1 - 2i$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$, which is the complex conjugate (that always happens for real matrices A).

   (c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \lambda_1 = -3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = -1, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \Rightarrow \lambda_1 = 2 + i, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 + i \end{bmatrix}$ and $\lambda_2 = 2 - i, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 - i \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
       For this one, we see that $\lambda = 1, 1$, but when we solve for the eigenvector, we just get $0 = 0$. That means that the values v_1, v_2 can each be any number (two free variables). Therefore, $\mathbf{v}_1, \mathbf{v}_2$ can be any vectors in the plane, as long as neither is zero and they are not multiples of each other.

(f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Similar to the last problem, $\lambda = 1, 1$, but there is only one eigenvector $v_2 = 0$, so $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if A is given below:

   (a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$
   SOLUTION: Substitute $x_2 = \frac{1}{2}(x_1' - x_1)$ into the second equation to get $x_1'' + 9x_1 = 0$, so
   $$x_1(t) = C_1 \cos(3t) + c_2 \sin(3t)$$
   Then put it back into the expression above to determine x_2:
   $$x_2(t) = \frac{3C_2 - C_1}{2} \cos(3t) - \frac{3C_1 + C_2}{2} \sin(3t)$$

   (b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
   SOLUTION: Substitute $x_2 = x_1' - x_1$ into the second equation, and find that $x_1'' - 2x_1' - 3x_1 = 0$. From that,
   $$x_1(t) = C_1 e^{-t} + C_2 e^{3t} \qquad x_2(t) = -2C_1 e^{-t} + 2C_2 e^{3t}$$

   (c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$
   SOLUTION: Note that its easier to start with the second equation rather than the first: $x_1 = -(x_2' + x_2)$. This gives us an IVP in x_2:
   $$x_2'' - 2x_2' + x_2 = 0 \quad \Rightarrow \quad x_2(t) = e^t(C_1 + C_2 t)$$
   Then
   $$x_1 = -2e^t(C_1 + C_2 t) - C_2 e^t$$

4. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the x_1, x_2 plane). Identify the origin as a *sink, source* or *saddle*:

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ The trace is 2, determinant is -24, so $\lambda^2 - 2\lambda - 24 = 0$ and $\lambda = -4, 6$.

   For $\lambda = -4$, we have $(1 - -4)v_1 + 5v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   For $\lambda = 6$, we have $(1 - 6)v_1 + 5v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   The general solution is:
   $$\mathbf{x}(t) = C_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   (b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ We should find $\lambda = 3, 5$.

   For $\lambda = 3$, we have $(7 - 3)v_1 + 2v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

   For $\lambda = 5$, we have $(7 - 5)v_1 + 2v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   $$\mathbf{x}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ We should find $\lambda = -2, 5$.

For $\lambda = -2$, we have $(2 - -2)v_1 + 3v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

For $\lambda = 5$, we have $(2 - 5)v_1 + 3v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 3 \\ -4 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(d) $A = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$ We should find $\lambda = -1, -2$.

For $\lambda = -1$, we have $(-1 + 1)v_1 + 0v_2 = 0$, so we just get $0 = 0$- Use the other equation to get $3v_1 + (-2 + 1)v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = -2$, we have $(-1 + 2)v_1 + 0v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (negative not necessary, but you can use it if you like)

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$