Review Questions, Exam 1, Math 244

These questions are presented to give you an idea of the variety and style of question that will be on the exam. It is not meant to be exhaustive, so be sure that you understand the homework problems and quizzes.

Discussion Questions:

1. True or False, and explain:

   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   False: This is an isocline associated with a slope of zero, and furthermore, $y = -2t$ is not a solution, and it is not a constant. Also, $y = -2t$ is not constant...

   (b) If $ty' + 2y = \cos(t)$, then the existence and uniqueness theorem will say that we have a unique solution to this DE with any initial value.

   False: In standard form, $p(t) = 2/t$ and $g(t) = \cos(t)/t$, both of which are not continuous at $t = 0$. If the initial condition is positive, we can say that there is a unique solution for $t > 0$ (similarly, if the initial time is negative, then the solution is valid for $t < 0$).

   (c) Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them (you may assume df/dy is continuous).

   True: If you draw a sketch in the (y, y') plane, we can see that f needs to have a third equilbrium, and the crossing would have to be from negative to positive (creating an unstable equilbrium). Of course, there may be more than that as well.

   (d) If $y' = \cos(y)$, then the solutions are periodic.

   False. If a function increases through, say, $y = k$, then it cannot decrease through $y = k$ at a later time since the DE is autonomous (slopes do not depend explicitly on time).

   (e) All autonomous equations are separable.

   True. Any autonomous equation can be written as $y' = f(y) \cdot 1$, which is separable and
   $\int \frac{dy}{f(y)} = \int dt.$

   (f) All separable equations are exact.

   True. If the equation is separable, then $y' = f(y)g(x)$, which can be written:

   $$\frac{1}{f(y)}\frac{dy}{dx} = g(x) \quad \Rightarrow \quad -g(x) + \frac{1}{f(y)}\frac{dy}{dx} = 0$$

   Now, if $M(x, y) = -g(x)$, then $M_y = 0$, and $N(x, y) = 1/f(y)$ means $N_x = 0$.

Solve:

1. $\frac{dy}{dx} = \frac{x^2 - 2y}{x}$

Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of x^2 to get:

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2. $(x+y)\,dx - (x-y)\,dy = 0$. Hint: Let $v = y/x$.

   Given the hint, rewrite the DE:

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+(y/x)}{1-(y/x)} = \frac{1+v}{1-v}$$

   With the substitution $xv = y$, we get the substitution for dy/dx:

$$v + xv' = y'$$

   So that the DE becomes:

$$v + xv' = \frac{1+v}{1-v} \qquad \Rightarrow \qquad xv' = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1+v} = \frac{1+v^2}{1-v}$$

   The equation is now separable:

$$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx \quad \Rightarrow \quad \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \ln|x| + C$$

   Therefore,

$$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$$

   Lastly, back-substitute $v = y/x$.

3. $\dfrac{dy}{dx} = \dfrac{2x+y}{3+3y^2-x} \qquad y(0) = 0.$

   This is exact. The solution is, with $y(0) = 0$,

$$-x^2 - xy + 3y + y^3 = 0$$

4. $\dfrac{dy}{dx} = -\dfrac{2xy+y^2+1}{x^2+2xy}$

   This is exact. The solution is: $x^2y + xy^2 + x = c$

5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0) = 2$

   This is linear and separable. $y(t) = \frac{2}{3}\sin(3t) + 2$, and the solution is valid for all time.

6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. State the interval on which the solution is valid.

   This is linear and separable. As a linear equation, the solution will be valid on all t (since $p(t) = -\frac{1}{2}$). The solution is $y(t) = 200\mathrm{e}^{(1/2)t}$

7. This is separable (or Bernoulli):

$$\int y^{-2}\,dy = \int (1-2x)\,dx \quad \Rightarrow \quad -\frac{1}{y} = x - x^2 + C$$

Put in the initial condition (IC): $6 = 0 + C$. Now finish solving explicitly:

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

The solution is valid on the interval $(-2, 3)$.

8. $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$

This is linear (but not separable). $y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{(1/2)t}$

9. $y' = \frac{1}{2}y(3-y)$

Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)}\,dy = \frac{1}{2}\int\,dt$$

Simplify your answer for y by dividing numerator and denominator appropriately to get:

$$y(t) = \frac{3}{(1/A)e^{-(3/2)t} + 1}$$

10. $\sin(2t)\,dt + \cos(3y)\,dy = 0$

Separable (and/or exact): $\frac{-1}{2}\cos(2t) + \frac{1}{3}\sin(3y) = C$

11. $y' = xy^2$

Separable: $y = \dfrac{1}{-(1/2)x^2 - C}$

12. $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{2x}{y}$ with substitution $v = y^2$.

SOLUTION: One method is to first multiply by y to get

$$yy' - \frac{3}{2x}y^2 = 2x$$

Let $v = y^2$, and therefore $v' = 2yy'$. Multiply by 2 to get the right form, then substitute

$$2yy' - \frac{3}{x}y^2 = 4x \quad \Rightarrow \quad v' - \frac{3}{x}v = 4x$$

Now it is a standard linear DE. Solving, we get $v = -4x^3 + Cx^3$, and

$$y^2 = -4x^3 + Cx^3$$

(We'll leave in implicit form).

13. $y' + 2y = g(t)$ with $y(0) = 0$ and $g(t) = 1$ on $0 \le t \le 1$ and zero elsewhere.

SOLUTION: This is similar to Exercise 33, Section 2.4. In this case, we go ahead and solve starting at time 0:

$$y' + 2y = 1 \quad \Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2t}$$

and this is valid for $0 \le t \le 1$. When we hit $t = 1$, the dynamics change to:

$$y' + 2y = 0 \quad \Rightarrow \quad y(t) = P\mathrm{e}^{-2t}$$

Now we will typically choose the constants so that y is continuous. Therefore, using our previous function, $y(1) = (1 - \mathrm{e}^{-2})/2$, and our current function: $y(1) = P\mathrm{e}^{-2}$, or

$$P = \frac{e^2 - 1}{2}$$

Therefore, the overall solution to the DE would be:

$$y(t) = \begin{cases} (1 - e^{-2t})/2 & \text{if } 0 \le t \le 1 \\ ((e^2 - 1)/2)\mathrm{e}^{-2t} & \text{if } 1 > 1 \end{cases}$$

Just for fun, the direction field and solution curve are plotted in Figure 1.

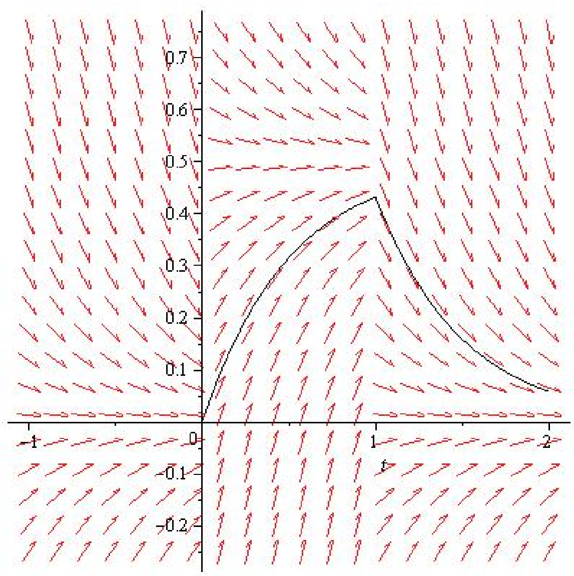


Figure 1: Direction field and solution curve for Exercise 14. Note how the solution approaches one equilibrium until $g(t)$ changes, then it goes to the new equilibrium.

Misc.

1. Show that $u(x,t) = \mathrm{e}^{-\alpha^2 t}\sin(x)$ is a solution to the PDE: $u_t = \alpha^2 u_{xx}$.

SOLUTION: Recall that when computing a partial derivative, all other variables are treated as a constant. Therefore,

$$u_t = -\alpha^2 \mathrm{e}^{-\alpha^2 t}\sin(x)$$

and the derivative with respect to x leaves the exponential alone and when we differentiate twice, a negative sign appears:

$$u_{xx} = -e^{-\alpha^2 t} \sin(x)$$

So by observation we see that $u_t = \alpha^2 u_{xx}$.

2. Determine all values of r for which $y = t^r$ is a solution to the DE: $t^2 y'' - 4ty' + 4y = 0$.

   SOLUTION: Substitute $y = t^r$, $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$ into the differential equation:

$$t^2(r(r-1)t^{r-2}) - 4t(rt^{r-1}) + 4t^r = 0 \quad \Rightarrow \quad t^r\left(r(r-1) - 4r + 4\right) = 0$$

   This equation should be true for all t, so we'll take out t^r and that leave the quadratic in r:

$$r^2 - 5r + 4 = 0 \quad \Rightarrow \quad (r-4)(r-1) = 0 \quad \Rightarrow \quad r = 1, 4$$

3. Write a DE of the form $y' = ay + b$ whose solutions all tend to $y = 3$.

   SOLUTION: There are a couple of ways to reason this out- Perhaps most straightforward is to sketch the line in the (y, y') plane. We want the line to go through the $y-$axis at $y = 3$ and the line should have negative slope. Choose a slope of -1 and the point $(3, 0)$ then the equation of the line is

$$y' - 0 = (-1)(y - 3) \quad \Rightarrow \quad y' = -y + 3$$

   ALTERNATE solution: We know the solution to $y' = ay + b$ is given by

$$Ce^{at} - \frac{b}{a}$$

   For this solution to converge to $y = 3$, $a < 0$ and $-b/a = 3$. One choice for that would be $a = -1$ and $b = 3$, which gives the same solution.

4. The population of mice ($P(t)$) in a field after t years satisfies the following IVP:

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{2500}\right) \qquad P(0) = 1000$$

   (a) What is the population when it is growing the fastest?

   SOLUTION: If we were to graph the population in the (P, P') plane, we would have an upside down parabola. The maximum rate of change is at the very top of the parabola, halfway between the two roots of $P = 0$ and $P = 2500$. Therefore, when $P = 1250$, the rate of change is the maximum.

   (b) What is the carrying capacity?

   SOLUTION: The carrying capacity is $P = 2500$.

   (c) Without solving the DE, what happens to our population as $t \to \infty$?

   SOLUTION: As with the first question, consider the upside down parabola in the (P, P') plane. Then for all initial $P > 0$, the population will tend to the threshold equilibrium at $P = 2500$. If the population begins with $P = 0$, then the population stays zero for all t. Finally, if the population begins as a negative number (not realistic, but it is part of our graph), then the population tends to negative infinity.

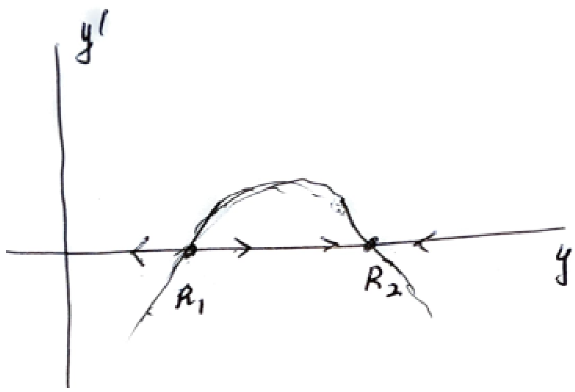
(d) What is the significance of the "3" in the equation?

SOLUTION: The 3 is the initial growth rate- That is, when P is very close to zero, the population grows exponentially like $P' = 3P$.

5. We want to construct a new population model. In this model, there is a "minimum population" so that, if the population ever goes below this number (call it a constant R_1), then there is not enough population to recover- The population goes extinct. There is also a larger number that we called the environmental threshold, R_2.

We want to find the simplest model that will have the desired behavior. First, noting that it is autonomous, graph $y' = f(y)$, then write down a possible polynomial for f.

SOLUTION: See the graph below (the idea is for you to sketch something similar in the (y, y') plane). The model has the form $y' = -(y - R_1)(y - R_2)$.



6. Suppose we have a tank that contains 100 L of water, in which there is 5 kg of salt. Brine is pouring into the tank at a concentration of 2 kg per liter, and at a rate of 4 liters per minute. The well mixed solution leaves the tank at a rate of 4 liters per minute.

Write the initial value problem that describes the amount of salt in the tank at time t, and solve.

SOLUTION: Recall that we want "Rate in − Rate out". In this case, let $Q(t)$ be the kg of salt in the tank at time t. Then:

$$\frac{dQ}{dt} = 2 \cdot 4 - 4 \cdot \frac{Q}{100}, \qquad Q(0) = 5$$

Solving, we would simplify to $Q' = -\frac{1}{25}Q + 8$. If you recall the solution to $y' = ay + b$, we can just write it down:

$$Q(t) = C\mathrm{e}^{-t/25} - 8/(-1/25) = C\mathrm{e}^{-t/25} + 200$$

With the initial condition, we get

$$Q(t) = -195\mathrm{e}^{-t/25} + 200$$

And we see that, as $t \to \infty$, then $Q(t) \to 200$ (which is 2 kg per liter, the same as the incoming concentration).

7. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank?

   SOLUTION:

   We see that, as $t \to \infty$, then $Q(t) \to 200$ (which is 2 kg per liter, the same as the incoming concentration).

8. Modify the tank problem so that the well mixed solution leaves the tank at a rate of 6 liters per minute. In this case, just write down the IVP, you don't need to solve it.

   SOLUTION: The incoming numbers did not change, just the outgoing. Notice that we will lose 2 L of fluid per minute, so the number of liters in the tank at time t will now be: $100 - 2t$. This gives:

$$\frac{dQ}{dt} = 8 - 6 \cdot \frac{Q(t)}{100 - 2t}, \qquad Q(0) = 5$$

9. Going back again to the original tank problem- If we change the rate at which the the well mixed solution leaves the tank to 2 liters per minute, what is the new IVP? (You don't need to solve the IVP).

   SOLUTION: The incoming numbers did not change, just the outgoing. Notice that we will gain 2 L of fluid per minute, so the number of liters in the tank at time t will now be: $100 + 2t$. This gives:

$$\frac{dQ}{dt} = 8 - 2 \cdot \frac{Q(t)}{100 + 2t}, \qquad Q(0) = 5$$

10. Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the environment (which we assume is some constant). Write down a differential equation which represents this statement, then find the general solution.

    SOLUTION: If $u(t)$ represents the temperature of the body, and T is the (constant) environmental temperature, then:

$$\frac{du}{dt} = -k(u - T)$$

    If you recall the solution to $y' = ay + b$, we can rewrite the equation to be: $u' = -ku + kT$, and then

$$u(t) = Ce^{-kt} - \frac{kT}{-k} \quad \Rightarrow \quad u(t) = Ce^{-kt} + T$$

11. Assume our ambient temperature is 20 degrees Celsius. My cup of coffee is initially at 50 degrees. Suppose we also know that after 2 minutes, the coffee is at 40 degrees. What equation would we need to solve in order to determine the minutes it takes for the coffee to become 30 degrees? (If we were allowed calculators, I'd ask you to solve it, but since we're not, just set up the equation).

    SOLUTION: We're given that $T = 2$, $u(0) = 50$ and $u(2) = 40$. Given the first piece of data, we can determine C, and with the second piece, we determine k:

$$u(0) = 50 \quad \Rightarrow \quad 50 = C \cdot 1 + 20 \quad \Rightarrow \quad C = 30$$

Now with the second piece of data, solve for k:

$$u(2) = 40 \quad \Rightarrow \quad 40 = 30\mathrm{e}^{-2k} + 20 \quad \Rightarrow \quad \frac{2}{3} = \mathrm{e}^{-2k}$$

Take the log of both sides and divide by -2: $k = -\frac{1}{2}\ln(2/3)$

Finally, with this value of k (we'll leave it as k in the equation), we can solve the following equation for t:

$$30 = 30\mathrm{e}^{-kt} + 20$$

We could solve this for t, but we were asked not to do so (for the sake of time).

12. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

    SOLUTION: Recall that, if $S(t)$ is the amount of money we will owe at time t, then

    $$\frac{dS}{dt} = rS - 12k \quad \Rightarrow \quad S(t) = C\mathrm{e}^{rt} + \frac{12k}{r}$$

    With the initial condition, we get

    $$S(t) = \left(10000 - \frac{12k}{r}\right)\mathrm{e}^{rt} + \frac{12k}{r}$$

    We substitute $r = 0.05$ and $t = 10$, then set this equation to 0 and solve for k:

    $$0 = \left(10000 - \frac{12k}{r}\right)\mathrm{e}^{10r} + \frac{12k}{r}$$

    Extra: If we go ahead and solve, we get a monthly payment rate of $k \approx \$105.90$.

13. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).

    SOLUTION: We'll try to solve it, and see what happens. The DE is separable. Notice that the function $f(x, y)$ from the E& U theorem is $(y - 1)/x$, which is not continuous at $x = 0$...

    Continuing as usual:

    $$\int \frac{1}{y-1}\,dy = \int \frac{1}{x}\,dx \quad \Rightarrow \quad \ln|y-1| = \ln|x| + C$$

    Exponentiate both sides to get

    $$y - 1 = Ax \quad \Rightarrow \quad y = Ax + 1$$

    There is no choice of A that will satisfy the initial condition,

    $$2 = A \cdot 0 + 1$$

14. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2).

SOLUTION: If $P(t)$ is the population at time t, then the first part of the statement translates to:

$$\frac{dP}{dt} = kP^{1/2}$$

where k is the constant of proportionality. This is separable (and autonomous), so:

$$\int P^{-1/2}\,dP = \int k\,dt \quad \Rightarrow \quad P^{1/2} = (k/2)t + C$$

Given the initial population, we can solve for C, given the second piece of info, we can solve for k:

$$P(0) = 20^2 \quad \Rightarrow \quad 20 = (k/2)(0) + C \quad \Rightarrow \quad C = 20$$

and $P(1) = 25^2$ gives:

$$25 = (k/2) + 20 \quad \Rightarrow \quad k = 10$$

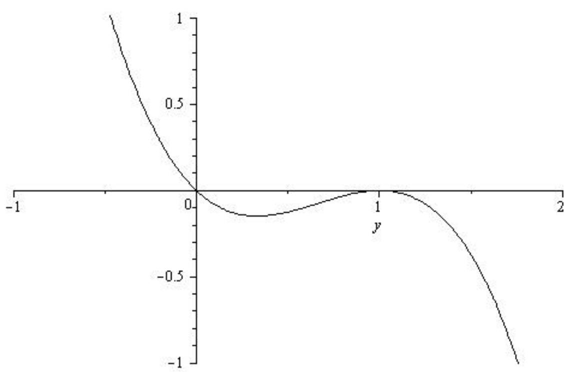
Therefore, our model is:

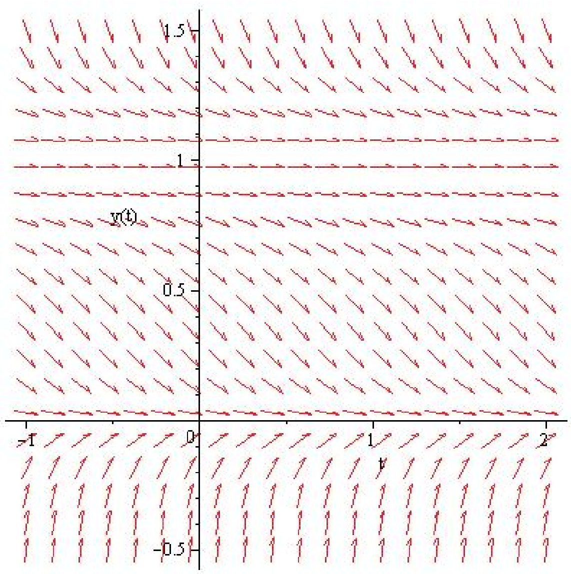
$$P(t) = (5t + 20)^2$$

Solving for the last part, $P(t) = 100^2$, we have

$$100 = 5t + 20 \quad \Rightarrow \quad t = 16$$

15. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.





SOLUTION: See the direction field above.

(a) Find and classify the equilibrium.

SOLUTION: From the sketch given, $y = 0$ is asymptotically stable and $y = 1$ is semistable.

(b) Find intervals (in y) on which $y(t)$ is concave up.

SOLUTION: Examine the intervals $y < 0$, $0 < y < 1/3$, $1/3 < y < 1$ and $y > 1$ separately. The function y will be concave up when dF/dy and F both have the same sign- This happens when F is either increasing and positive (which happens nowhere) or decreasing and negative:

$$0 < y < \frac{1}{3} \qquad y > 1$$

(c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. See the figure above.

(d) Find an appropriate polynomial for $F(y)$.

SOLUTION: One example is

$$y' = -y(y-1)^2$$

16. Consider the IVP: $t(t-5)y' + \cos(t)y = t^2$, with $y(1) = 2$. Using the existence and uniqueness theorem, does there exist a unique solution to the IVP? If so, what is the full interval on which it exists?

SOLUTION: The functions p, g in the standard form of a linear DE would be continuous for all t except $t = 0$ adn $t = 5$, which cuts t into three intervals. We choose the one with the initial t, in this case, $t = 1$, so that our interval will be $(0, 5)$.

17. Given the DE below, state where in the $ty-$plane the hypotheses of the existence and uniqueness theorem are satisfied.

$$y' = \frac{1+t^2}{3y - y^3}$$

SOLUTION: If $y' = f(t, y)$, then we need both f and $\partial f/\partial y$ to be continuous. In this case,

$$f(t, y) = \frac{1+t^2}{3y-y^3} = (1+t^2)(3y-y^3)^{-1}$$

so that

$$\frac{\partial f}{\partial y} = (1+t^2)(-(3y-y^3)^{-2}(3-3y^2)$$

In both cases, we want to avoid the denominator being zero. Therefore, the hypotheses for the existence and uniqueness theorem are satisfied everywhere in the (t, y) plane except for the vertical lines

$$y = 0, \qquad y = \sqrt{3}, \qquad y = -\sqrt{3}$$

18. Given the direction field below, find a differential equation that is consistent with it.

SOLUTION: Draw the corresponding figure in the (y, y') plane first. There we see that $y = 0$ is unstable (make it a linear crossing), and $y = 2$ is stable, $y = 4$ is unstable. From the figure,

$$y' = y(y-2)(y-4)$$

will work.

19. Consider the direction field below, and answer the following questions:

   (a) Is the DE possibly of the form $y' = f(t)$?
   SOLUTION: No. The isoclines would be vertical (consider, for example, a vertical line at $t = -3$; the slopes are clearly not equal).

   (b) Is the DE possible of the form $y' = f(y)$?
   SOLUTION: No. The isoclines would be horizontal (for example, look at a horizontal line at $y = 1$- Some slopes are zero, others are not).

   (c) Is there an equilibrium solution? (If so, state it):
   SOLUTION: Yes- At $y = 0$.

   (d) Draw the solution corresponding to $y(-1) = 1$.
   SOLUTION: Just draw a curve consistent with the arrows shown.

20. Evaluate the following integrals:

$$\int x^3 e^{2x}\,dx \qquad \int \frac{x}{(x-1)(2-x)}\,dx \qquad e^{-\int 3/t\,dt}$$

SOLUTIONS:

$$\int x^3 e^{2x}\,dx$$

sign	u	dv
$+$	x^3	e^{2x}
$-$	$3x^2$	$(1/2)e^{2x}$
$+$	$6x$	$(1/4)e^{2x}$
$-$	6	$(1/8)e^{2x}$
$+$	0	$(1/16)e^{2x}$

Therefore, the integral is:

$$e^{2x}\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right) + C$$

In the second integral, we need partial fractions first. I'm also pulling out a negative sign to make the second factor $(x-2)$:

$$\frac{-x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{1}{x-1} - \frac{2}{x-2}$$

Therefore,

$$\int \frac{x}{(x-1)(2-x)}\,dx = \ln|x-1| - 2\ln|x-2| + C$$

For the third integral:

$$e^{-3\ln(t)} = e^{\ln(t^{-3})} = t^{-3}$$