Sections 3.8

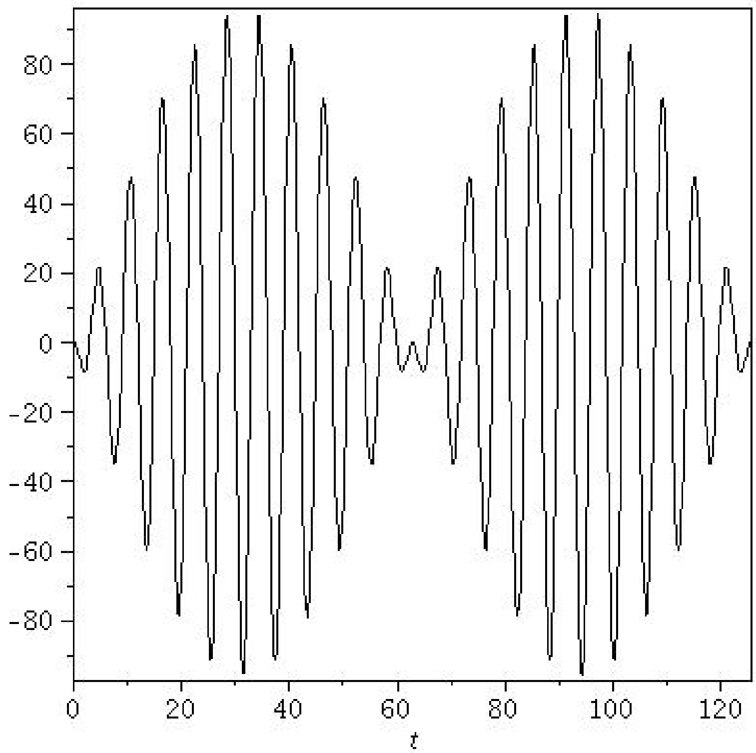
Last time, we focused our attention on the spring-mass model that had no forcing. Today, we look at the special case of **periodic forcing.**

Adding wave forms together

What happens when we subtract (or add) two waves together of slightly different circular frequency. Here, we're thinking of $\omega \approx \omega_0$. Using a trig identity, the difference can be written as a product of sines:

$$\cos(\omega t) - \cos(\omega_0 t) = 2 \sin\left(\frac{\omega + \omega_0}{2}\, t\right) \sin\left(\frac{\omega - \omega_0}{2}\, t\right)$$

The graph has an "envelope" with the larger period, and a faster moving wave inside of it.



This phenomenon is known as **beating**. The envelope uses the sine with the difference of ω's:

$$\sin\left(\frac{\omega - \omega_0}{2} t\right)$$

In the figure, $\omega = 1$ and $\omega_0 = 1.1$. The overall period would be

$$\frac{2 \cdot 2\pi}{0.1} \approx 125.66$$

But we see that the **period of one beat** would actually be half that, $2\pi/|\omega - \omega_0| \approx 60$. In general, we see that the **circular frequency of one beat** is $|\omega - \omega_0|$.

Back to our Model: No Damping, Periodic Forcing

Rather than write the model as $mu'' + ku = F(t)$, it is common practice to write the model as $u'' + \omega_0^2 u = F(t)$. That way it is easy to read off the circular frequency of the homogenous part of the solution. Putting in the periodic forcing (we'll use cosine)

$$u'' + \omega_0^2 u = F_0 \cos(\omega t)$$

And, as long as the forcing function does not have the same frequency as the natural frequency, $\omega_0 \neq \omega$, so the overall solution is:

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{w_0^2 - w^2} \cos(wt)$$

It can be shown that, if we use zero initial conditions $u(0) = u'(0) = 0$, then

$$C_1 = -\frac{F_0}{\omega_0^2 - \omega^2} \qquad C_2 = 0$$

In this case,

$$u(t) = \frac{F_0}{\omega_0^2 - \omega^2} \left(\cos(wt) - \cos(w_0 t)\right)$$

Therefore, **when** $\omega \approx \omega_0$, we get a phenomena known as **beating**, which occurs when two waveforms of almost the same frequency are added (or subtracted).

From the first part of the notes, we see that the circular frequency of one beat is $|\omega - \omega_0|$. However, also note what happens with the **amplitude**. Since $\omega_0^2 - \omega^2 \to 0$, the amplitude of the beating goes to infinity, and this causes what is known as **resonance**.

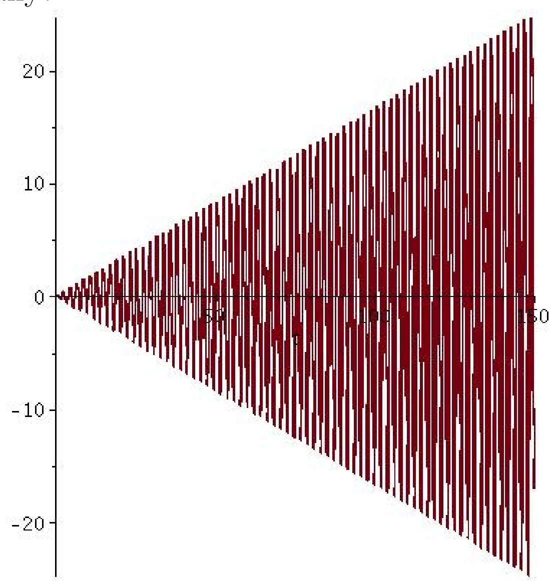
Resonance

Resonance occurs when the natural frequency and the forcing frequency match: $\omega = \omega_0$. What does this look like graphically and algebraically?

- Graphically, we see that the period of the larger beat becomes infinitely long, with an infinitely large amplitude.
- Algebraically, we can find the solution using l'Hospital's rule:

$$\lim_{w \to w_0} \frac{F_0 \left(\cos(wt) - \cos(w_0 t)\right)}{w_0^2 - w^2} =$$

$$\lim_{\omega \to \omega_0} \frac{-F_0 t \sin(\omega t)}{-2\omega} = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$



We could have solved this using the Method of Undetermined Coefficients as well, and we would have multiplied the ansatz by t.

Bringing in Damping: The Full Model

When we have the full damped system, then all solutions will tend to zero as $t \to \infty$. How do we know that? Given

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

We will never have purely periodic solutions for y_h, therefore, y_p will never have the multiplication by t (by undetermined coefficients). HOWEVER, we will still analyze a slightly changed version of the system:

$$u'' + pu' + qu = \cos(\omega t)$$

It takes a bit of work, but it can be shown that the **particular solution** can be expressed as

$$R\cos(\omega t - \delta)$$

where (we won't use δ):

$$R = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}$$

The key question:

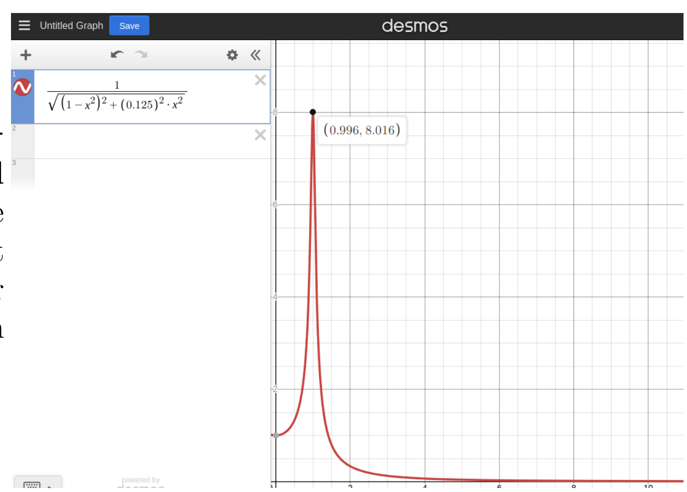
Is there a value of ω that **maximizes** the amplitude, R, for the particular solution?

To give an example, the example IVP in the text is

$$u'' + 0.125u' + u = \cos(\omega t), \qquad u(0) = 2, u'(0) = 0$$

Using our formula for the amplitude of the particular solution, $p = 0.125$ and $q = 1$ so

$$R = \frac{1}{\sqrt{(1 - \omega^2)^2 + (0.125)^2\omega^2}}$$



We might expect a maximum amplitude at approximately $\sqrt{k/m}$, which in this case is 1, and that is true. From the graph, we can determine that of all possible values of ω, the one that gives a maximum amplitude to the particular solution is when $\omega \approx 0.996$ and the maximum amplitude is a little over 8.

To find the maximum algebraically, we'll differentiate R with respect to ω (or whatever variable we want to optimize), then set the derivative to zero. Before we do that, notice the following "shortcut":

$$R = \frac{1}{\sqrt{f(\omega)}} = (f(\omega))^{-1/2} \quad \Rightarrow \quad \frac{dR}{d\omega} = -\frac{1}{2}(f(\omega))^{-3/2} f'(\omega) = -\frac{1}{2}\,\frac{f'(\omega)}{(f(\omega))^{3/2}}$$

If we set this to zero,

$$\frac{dR}{d\omega} = 0 \quad \Rightarrow \quad -\frac{1}{2}\,\frac{f'(\omega)}{(f(\omega))^{3/2}} = 0 \quad \Rightarrow \quad f'(\omega) = 0$$

This simplifies things a lot- Given $R = f(\omega)^{-1/2}$, to find where $dR/d\omega = 0$, we simply need to find where $f'(\omega) = 0$. Doing that:

$$f(\omega) = (q - \omega^2)^2 + p^2\omega^2 \quad \Rightarrow \quad \frac{df}{d\omega} = 2(q - \omega^2)(-2\omega) + p^2 \cdot 2\omega = 0$$

Solving for only the positive ω, we get $\omega = \sqrt{\frac{2q - p^2}{2}}$.

As an example, what is the value of ω that gives the maximum amplitude in our previous example,

$$u'' + 0.125u' + u = \cos(\omega t)$$

The value of ω is:

$$\sqrt{\frac{2(1) - 0.125^2}{2}} \approx 0.996087$$

Therefore, if we "tune" the external forcing function to have that circular frequency, we will maximize the amplitude of the particular solution.

Homework
Replaces Section 3.8

1. Solve the IVP $u'' + \omega_0^2 u = F_0 \cos(\omega t)$, $u(0) = 0$ and $u'(0) = 0$, if $\omega \neq \omega_0$.
2. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$.

   NOTE: We defined L to be the length of the spring stretched from its natural length to equilibrium.
3. Convert the following to $R\cos(\omega_0 t - \delta)$.

   (a) $\cos(9t) - \sin(9t)$
   (b) $2\cos(3t) + \sin(3t)$
   (c) $-2\pi\cos(\pi t) - \pi\sin(\pi t)$
   (d) $5\sin(t/2) - \cos(t/2)$
4. Suppose $u'' + 4u = \cos(2.8t)$. This function exhibits beating. (i) Give the frequency of a beat, and (ii) Give (only) the particular part of the solution.
5. Same question as before, but with $u'' + 9u = \cos(3.1t)$.
6. Same question as before, but with $u'' + u = \cos(1.3\,t)$
7. Find the solution to $u'' + 9u = \cos(3t)$ with zero initial conditions.
8. Find the particular solution: $y'' + 3y' + 2y = \cos(t)$.
9. Consider $u'' + pu' + qu = \cos(\omega t)$. In the notes at the bottom of p. 4, we got that

   $$\omega = \sqrt{\frac{2q - p^2}{2}}$$

   Thinking of p as damping, if the damping is very very small, then approximately what value of ω will result in a very large amplitude response?
10. Consider $u'' + u' + 2u = \cos(\omega t)$. In our notes, we said that we could show that the amplitude of the particular solution is given by

   $$R = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}$$

   Find the value of ω that will maximize the amplitude of the particular solution.
11. Suppose we can tune the value of q rather than the value of ω in the differential equation (where $\omega = 3$):

   $$u'' + u' + qu = \cos(3t)$$

   Using the R from the last question, find the value of q that will maximize the amplitude of the particular soluton.