Solutions to the Homework
Replaces Section 3.8

1. Solve the IVP $u'' + \omega_0^2 u = F_0\cos(\omega t)$, $u(0) = 0$ and $u'(0) = 0$, if $\omega \neq \omega_0$.

   SOLUTION: If $\omega \neq \omega_0$, it's pretty straightforward:

   $$u_h(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t), \qquad u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

   You should be able to find that the full solution (with zero initial conditions) is

   $$u(t) = \frac{F_0}{\omega_0^2 - \omega^2}(\cos(\omega\, t) - \cos(\omega_0\, t))$$

2. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$

   SOLUTION: With no damping, $mu'' + ku = 0$ has solution

   $$u(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

   so the period is given below. We also note that $mg - kL = 0$, and this equation yields the desired substitution:

   $$P = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{and} \qquad mg = kL \quad \Rightarrow \quad \frac{k}{m} = \frac{g}{L}$$

3. Convert the following to $R\cos(\omega t - \delta)$

   (a) $\cos(9t) - \sin(9t)$
   In this case, $R = \sqrt{2}$ and $\delta = \tan^{-1}(-1) = -\pi/4$
   Note that in this case, we don't need to add π because $(1, -1)$ is in Quadrant IV.

   (b) $2\cos(3t) + \sin(3t)$
   SOLUTION: $R = \sqrt{5}$ and $\omega = 3$. The angle δ is computed as the argument of the point $(2, 1)$, which you can leave as $\delta = \tan^{-1}(1/2)$:

   $$2\cos(3t) + \sin(3t) = \sqrt{5}\cos(3t - \tan^{-1}(1/2))$$

   (c) $-2\pi\cos(\pi t) - \pi\sin(\pi t)$
   SOLUTION: Same idea, but note that $(-2\pi, -\pi)$ is a point in Quadrant III, so we add (or subtract) π:

   $$R = \pi\sqrt{5} \qquad \text{and} \qquad \delta = \tan^{-1}(1/2) + \pi \qquad \text{and} \qquad \omega = \pi$$

   (d) $5\sin(t/2) - \cos(t/2)$
   SOLUTION: Did you notice I reversed the sine and cosine on you (that was a mistake, but maybe it was a helpful one). The value of R and ω would be the same either way, but δ changes:

$$R = \sqrt{26} \qquad \omega = \frac{1}{2}$$

For δ, notice that our "point" is $(-1, 5)$ which is in Quadrant II, so add π:

$$\sqrt{26}\cos\left(\frac{t}{2} - \tan^{-1}(-5) - \pi\right)$$

4. $u'' + 4u = \cos(2.8\,t)$

   TOLUTION: The (circular) frequency of the beats is $|\omega_0 - \omega|$, or in this case, 0.8 or 4/5. The particular part of the solution is

$$\frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t) = \frac{1}{3.84}\cos(2.8\,t) \approx 0.26\cos(2.8\,t)$$

5. $u'' + 9u = \cos(3.1\,t)$

   For this, the (circular) beat frequency is $|\omega_0 - \omega| = 1/10$. The particular part of the solution is

$$\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t) = -1.64\cos(3.1t)$$

6. $u'' + u = \cos(1.3\,t)$

   For this, the (circular) beat frequency is 3/10. The particular part of the solution is

$$\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t) = -1.45\cos(1.3t)$$

7. Solve $u'' + 9u = \cos(3t)$ with zero ICs.

   The solution is:

$$u(t) = \frac{1}{6}\,t\,\sin(3t)$$

8. Find the particular solution of the given differential equation:

$$y'' + 3y' + 2y = \cos(t)$$

   SOLUTION: Using the Method of Undetermined Coefficients, we have

$$y_p = A\cos(t) + B\sin(t)$$

   And substitute into the DE to solve for A, B. It takes a little bit of bookkeeping, but we should get:

$$\frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$$

9. Consider $u'' + pu' + qu = \cos(\omega t)$. In the notes at the bottom of p. 4, we got that

$$\omega = \sqrt{\frac{2q - p^2}{2}}$$

   Thinking of p as damping, if the damping is very very small, then approximately what value of ω will result in a very large amplitude response?

   SOLUTION: If the damping is very small, then the maximizer ω becomes very close to $\sqrt{q}$, which is what we would expect from no damping (and then resonance).

10. Consider $u'' + u' + 2u = \cos(\omega t)$. Find the value of ω that will maximize the amplitude of the response.

    NOTE: I don't want you to memorize the value of ω. Rather, find the amplitude R, then differentiate to find where the derivative is zero. Remember our shortcut (dealing with $f(\omega)$).

    SOLUTION: Differentiate the given R with respect to ω (remember the shortcut):

$$R = \frac{1}{\sqrt{(2-\omega^2)^2 + \omega^2}}$$

    We looked at a shortcut for differentiating this and setting it to zero- That's the same as just differentiating $(2-\omega^2)^2 + \omega^2$ and setting that to zero.

    Doing that, we get $\omega = \sqrt{6}2 \approx 1.22$.

11. Suppose we can tune the value of q rather than the value of ω in the differential equation (where $\omega = 3$):

$$u'' + u' + qu = \cos(3t)$$

    Find the value of q that will maximize the amplitude of the forced response.

    SOLUTION: Using the formula for R that would be given (as in the previous problem),

$$R = \frac{1}{\sqrt{(q-9)^2 + 9}}$$

    To find the q that maximizes R, differentiate and set to zero. As before, we can use a shortcut:

$$\frac{d}{dq}\left[(q-9)^2 + 9\right] = 0 \qquad \Rightarrow \qquad q = 9$$