Solutions - Homework (to replace 7.2)

1. Let A, B be the matrices below. Compute the matrix operation listed.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

   (a) $2A + B$
   SOLUTION:

$$\begin{bmatrix} 4 & -5 \\ 3 & 7 \end{bmatrix}$$

   (b) AB

$$\left[\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

   (c) BA

$$\left[\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ -7 & 5 \end{bmatrix}$$

   (d) $A^T + B^T$

$$\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & 4 \end{bmatrix}$$

   (e) A^{-1} The formula for the inverse is:

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$$

   (f) B^{-1}
   Using the previous formula,

$$B^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

2. Vectors and matrices might have complex numbers. If $z = 3 + 2i$ and vector $\mathbf{v} = [1 + i, 2 - 2i]^T$, then find the real part and the imaginary part of $z\mathbf{v}$.
   SOLUTION:

$$z\mathbf{v} = (3 + 2i) \begin{bmatrix} 1 + i \\ 2 - 2i \end{bmatrix} = \begin{bmatrix} (3 + 2i)(1 + i) \\ (3 + 2i)(2 - 2i) \end{bmatrix} = \begin{bmatrix} 1 + 5i \\ 10 - 2i \end{bmatrix}$$

   Therefore,

$$\text{real}(z\mathbf{v}) = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \qquad \text{imag}(z\mathbf{v}) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

3. What will the graph of $e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be (where t is any real number).

   SOLUTION: This is the set of all (positive) multiples of the vector, so it is a "ray" extending through the origin and through $(1, 2)$, and then outward.

4. Adding two vectors: Geometrically (and numerically) compute the following, where $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Be sure to draw each vector out, and see if you can see a pattern.

   (a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} - 2\mathbf{v}$ (c) $\mathbf{u} + \frac{1}{2}\mathbf{v}$ (d) $-\mathbf{u} + \mathbf{v}$

   SOLUTION: See the figures attached. The "rule" is a parallelogram rule- Do you see the parallelograms?

5. Verify that $\mathbf{x}_1(t)$ below satisfies the DE below.

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   SOLUTION: Let $x_1(t) = e^{3t}$ and $x_2(t) = 2e^{3t}$. Then verify that:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$$

6. Consider
$$\begin{aligned} x' &= 2x + 3y + 1 \\ y' &= x - y - 2 \end{aligned}.$$

   First find the equilibrium solution, x_e, y_e.

   SOLUTION: The equilibrium is where $x' = y' = 0$, or

$$\begin{aligned} 2x + 3y &= -1 \\ x - y &= 2 \end{aligned} \quad \Rightarrow \quad x_e = 1, y_e = -1$$

   Then show that, if $u = x - x_e$ and $v = y - y_e$, then

$$\begin{aligned} u' &= 2u + 3v \\ v' &= u - v \end{aligned}$$

   SOLUTION: We're making a change of variables, $u = x - 1$ and $v = y + 1$. Then $u' = x'$ and $v' = y'$. Now, for the other side of the equations,

$$\begin{aligned} 2u + 3v = 2(x - 1) + 3(y + 1) = 2x + 3y + 1 \\ u - v = (x - 1) - (y + 1) = x - y - 2 \end{aligned}$$

   which are the original ODEs.

7. Each system below is *nonlinear*. Solve each by first writing the system as dy/dx.

(a) $\begin{aligned} x' &= y(1+x^3) \\ y' &= x^2 \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \quad \Rightarrow \quad y\,dy = \frac{x^2}{1+x^3}\,dx$

Now,

$$\frac{1}{2}y^2 = \frac{1}{2}\ln(1+x^3) + C$$

(b) $\begin{aligned} x' &= 4+y^3 \\ y' &= 4x-x^3 \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4x-x^3}{4+y^3} \quad \Rightarrow \quad (4+y^3)\,dy = 4x - x^3\,dx$

Now,

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$$

(c) $\begin{aligned} x' &= 2x^2y+2x \\ y' &= -(2xy^2+2y) \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-(2xy^2+2y)}{2x^2y+2x}$

$$(2xy^2+2y) + (2x^2y+2x)\frac{dy}{dx} = 0$$

This is exact, with $M_y = 4xy+2$ and $N_x = 4xy+2$.
Antidifferentiating M with respect to x, we get:

$$x^2y^2 + 2xy$$

If we differentiate that with respect to y, we get $2x^2y+2x$, so that's our function.
The solution is then:

$$x^2y^2 + 2xy = C$$