Example: Algebra on the Index

Rewrite the sum so that the generic term involves x^n:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

SOLUTION: Let $n = m - 2$ (so that $m = n + 2$). Make the substitutions:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Example: Algebra on the Index

Simplify to one sum that uses the term x^n:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=?}^{?} \left(\quad C_n \quad \right) x^n$$

SOLUTION: Try writing out the first few terms:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 2\cdot 3a_3 x + 3\cdot 4a_4 x^2 + 4\cdot 5a_5 x^3 + \cdots$$

$$\sum_{k=1}^{\infty} k a_k x^k = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

Powers of x don't line up: *To write this as a single sum, we need to manipulate the sums so that the powers of x line up.*

SOLUTION 1

Pad the second equation by starting at $k = 0$:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_5 x^3 + \cdots$$

$$\sum_{k=0}^{\infty} k a_k x^k = 0 + a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

Substitute $n = m - 2$ (or $m = n + 2$) into the first sum, and $n = k$ into the second sum:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\sum_{k=0}^{\infty} k a_k x^k = \sum_{n=0}^{\infty} n a_n x^n$$

Finishing the solution:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n$$

$$= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + n a_n\right)x^n$$

Alternate Solution:

We could have started both indices using x^1 instead of x^0.
Here are the sums again:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \left[3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \cdots\right]$$

$$\sum_{k=1}^{\infty} k a_k x^k = \qquad a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

In this case,

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \sum_{m=3}^{\infty} m(m-1)a_m x^{m-2}$$

and let $n = m - 2$ to get

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + n a_n\right] x^n$$

Using Series in DEs

Given $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = y_0$ and $y'(x_0) = v_0$, assume y, p, q are analytic at x_0.

Ansatz:

$$y(t) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

so that

$$y'(t) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \quad \text{and} \quad y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

The Big Picture

Substituting the series into the DE will give something like:

$$\sum(\ldots) + \sum(\ldots) + \sum(\ldots) = 0$$

We will want to write this in the form:

$$\sum(\quad C_n \quad)x^n = 0$$

Then we will set $C_n = 0$ for each n.

Example: Power Series into a DE

Find the recurrence relation and the first four terms of a fundamental set of solutions to:

$$y'' - xy' - y = 0 \qquad x_0 = 1$$

Substitute $y = \sum_{n=0}^{\infty} a_n (x-1)^n$:

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Look at/Simplify the second term:

$$-x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} =$$

$$-x\sum_{n=1}^{\infty} na_n(x-1)^{n-1}+1\sum_{n=1}^{\infty} na_n(x-1)^{n-1}-1\sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

to get something useful:

$$-\sum_{n=1}^{\infty} na_n(x-1)^{n} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

We now have 4 sums:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} na_n(x-1)^n$$

$$-\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Ready to simplify? (Check powers)

Reset the indices:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} na_n(x-1)^n$$

$$-\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Make the substitutions:

$$\begin{array}{llll} k = n-2 & k = n & k = n-1 & k = n \\ n = k+2 & n = k & n = k+1 & n = k \end{array}$$

To get:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k\right)(x-1)^k = 0$$

Principle: If $P(x) = 0$ for all x, then the coefficients of the polynomial are all zero.
Therefore,

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k\right)(x-1)^k = 0$$

means:

$$((k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k = 0$$

Solve for a_{k+2}:

$$a_{k+2} = \frac{1}{k+2}\left(a_k + a_{k+1}\right)$$

This is the recurrence relation.

Build up a solution from the recurrence

Find y_1, y_2 that solve:

$y(1) = a_0 = 1 \quad y'(1) = a_1 = 0 \qquad y(1) = a_0 = 0 \quad y'(1) = a_1 = 1$

(Therefore, $W(y_1, y_2)(1) = 1$)
We do this for y_1 in the first column, y_2 in the second:

$$\begin{aligned} k=0 &\quad a_2 = \tfrac{1}{2}(a_0 + a_1) = \tfrac{1}{2} \\ k=1 &\quad a_3 = \tfrac{1}{3}(a_1 + a_2) = \tfrac{1}{6} \\ k=2 &\quad a_4 = \tfrac{1}{4}(a_2 + a_3) = \tfrac{1}{6} \\ k=3 &\quad a_5 = \tfrac{1}{5}(a_3 + a_4) = \tfrac{1}{15} \end{aligned} \quad \text{and} \quad \begin{aligned} k=0 &\quad a_2 = \tfrac{1}{2}(a_0 + a_1) = \tfrac{1}{2} \\ k=1 &\quad a_3 = \tfrac{1}{3}(a_1 + a_2) = \tfrac{1}{2} \\ k=2 &\quad a_4 = \tfrac{1}{4}(a_2 + a_3) = \tfrac{1}{4} \\ k=3 &\quad a_5 = \tfrac{1}{5}(a_3 + a_4) = \tfrac{3}{20} \end{aligned}$$

We now have two linearly independent solutions to the DE:

$$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \dots$$

and

$$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots$$