Selected solutions, Section 5.2

We'll just get the recurrence relation for the power series solutions to 1, 3, 4, 6 and 7.

1. $y'' - y = 0$, $x_0 = 0$.

   SOLUTION: Use the ansatz- in this case, based at $x_0 = 0$, we use: $y = \sum_{n=0}^{\infty} a_n x^n$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

   Substitution into the DE,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

   To make this into a single sum, be sure your series all begin with the same power (in this case, the first term of both sums has x^0). Substitute for each index- in the first sum, let $m = n - 2$ (or $n = m + 2$), and $m = n$. Then:

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

   Now we can make this into a single sum:

$$\sum_{m=0}^{\infty} ((m+2)(m+1)a_{m+2} - a_m)x^m = 0$$

   Each coefficient must be zero, which gives us the recurrence relation (solve for a_{m+2}):

$$(m+2)(m+1)a_{m+2} - a_m = 0 \quad \Rightarrow \quad a_{m+2} = \frac{a_m}{(m+1)(m+2)}, \text{ for } m = 0, 1, 2, 3, \ldots$$

3. $y'' - xy' - y = 0$, $x_0 = 1$.

   Ansatz: $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, so that the derivatives are $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2}$.

   The differential equation becomes:

$$\sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} - x\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

   In the middle expression, we need to incorporate the x into the sum. Since our sum is in powers of $x - 1$, we'll re-write $x = ((x-1)+1)$, so just working with this sum, we have:

$$x\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = ((x-1)+1)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

Now we have 4 sums to make into one sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

We note that all the sums except for the second have a first term $(x-1)^0$. So we can change the index of the second sum to start at $n=0$ instead of $n=1$ (the first term will be $0(x-1)^0$, so we're not changing the sum). Doing that,

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Unify the index so that all have $(x-1)^m$. For each sum above, we make the following substitutions (respectively):

$$\begin{array}{cccc} m = n-2 & & m = n-1 & \\ n = m+2 & m = n & n = m+1 & m = n \end{array}$$

Now the sums can be combined:

$$\sum_{m=0}^{\infty} \left((m+2)(m+1)a_{m+2} - ma_m - (m+1)a_{m+1} - a_m\right)(x-1)^m = 0$$

This equation holds true for all x in the radius of convergence, so each of the coefficients must be zero.

$$(m+2)(m+1)a_{m+2} - ma_m - (m+1)a_{m+1} - a_m = 0$$

Solving for a_{m+2} gives us the recurrence relation:

$$a_{m+2} = \frac{(m+1)a_{m+1} + (m+1)a_m}{(m+1)(m+2)} = \frac{a_{m+1} + a_m}{m+2}, \text{ for } m = 0, 1, 2, \ldots$$

4. $y'' + k^2x^2y = 0$, $x_0 = 0$, k is a constant.

Ansatz: $y = \sum_{n=0}^{\infty} a_n x^n$, and we only need y'' in this case, where

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

The differential equation becomes:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + k^2x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Incorporate the k^2x^2 into the second sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

To combine sums, the starting powers should be the same. They are not, since the first sum starts with x^0 and the second with x^2. The complication here is that we can't simply start the second index at a smaller number. However, consider writing out the terms of the first sum:

$$2 \cdot 1 \cdot a_2 x^0 + 3 \cdot 2 \cdot a_3 x^1 + 4 \cdot 3 \cdot a_4 x^2 + 5 \cdot 4 \cdot a_5 x^3 + \cdots$$

Therefore we can rewrite the first sum as:

$$2a_2 + 6a_3 x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2}$$

And now the first power of the sum is x^2, as desired. Substitute this into the first sum to get:

$$2a_2 + 6a_3 x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

Now, in the first sum, let $m = n-2$ (or $n = m+2$), and in the last sum, let $m = n+2$ (or $m = m-2$). Now we can combine sums:

$$2a_2 + 6a_3 x + \sum_{m=2}^{\infty} ((m+2)(m+1)a_{m+2} + k^2 a_{m-2})x^m = 0$$

Now we set the coefficients to zero, as usual. From the first two terms, we see that

$$a_2 = 0, \quad a_3 = 0$$

and for all other terms:

$$a_{m+2} = \frac{-k^2}{(m+2)(m+1)} a_{m-2} \text{ for } m = 2, 3, 4, \ldots$$

6. $(2+x^2)y'' - xy' + 4y = 0$, $x_0 = 0$.

Ansatz: $y = \sum_{n=0}^{\infty} a_n x^n$, so that the derivatives are

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

The differential equation becomes:

$$(2+x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

The first two sums need some work- let's start by incorporating $(2+x^2)$ into the first sum:

$$(2+x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

We might as well make every power start with x^0 while we're at it, so the index of the last sum above can be written as $n = 0$ rather than $n = 2$:

$$(2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

In the middle expression, we need to incorporate the x into the sum. In this case, it's straightforward since we have powers of x (and we'll go ahead and make the sum begin with x^0):

$$x\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n$$

Now we have 4 sums to make into one sum:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

In the first sum, let $m = n - 2$ (or $n = m + 2$), and for the others, $m = n$. Then combining sums gives us:

$$\sum_{m=0}^{\infty} \left(2(m+2)(m+1)a_{m+2} + \left(m(m-1) - m + 4\right)a_m\right)x^m = 0$$

Solve for the recurrence relation:

$$a_{m+2} = \frac{m^2 - 2m + 4}{2(m+2)(m+1)}a_m \qquad \text{for } m = 0, 1, 2, \ldots$$

7. $y'' + xy' + 2y = 0$, $x_0 = 0$.

   Ansatz: $y = \sum_{n=0}^{\infty} a_n x^n$, so that the derivatives are

   $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

   The differential equation becomes:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

In the middle expression, we need to incorporate the x into the sum. In this case, it's straightforward since we have powers of x (and we'll go ahead and make the sum begin with x^0):

$$x\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n$$

With that, we can combine the three sums:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

In the first sum, let $m = n - 2$ (or $n = m + 2$), and for the others, $m = n$. Then combining sums gives us:

$$\sum_{m=0}^{\infty} \left((m+2)(m+1)a_{m+2} + (m+2)a_m\right) x^m = 0$$

Solve for the recurrence relation:

$$a_{m+2} = -\frac{1}{m+1} a_m \qquad \text{for } m = 0, 1, 2, \ldots$$

For problems 15, 17, 18 below, the textbook had you going back to the recurrence relation to find the terms of the sum, but we want to practice using the derivatives to find the terms of the sum.

We determine the first five non-zero terms in the solution by taking the derivatives, where we use the relationship between the coefficients and the derivatives:

$$a_n = \frac{y^{(n)}(x_0)}{n!}$$

15. $y'' - xy' - y = 0$ with $y(0) = 2, y'(0) = 1$.

Solution:

From the initial conditions, $a_0 = 2$ and $a_1 = 1$. Now find $y''(0)$:

$$y'' = xy' + y \quad \Rightarrow y''(0) = 0 \cdot y'(0) + y(0) \quad \Rightarrow \quad y''(0) = y(0) = 2$$

Now $y'''(0)$:

$$y''' = y' + xy'' + y' = 2y' + xy'' \quad \Rightarrow \quad y'''(0) = 2 \cdot y'(0) = 2$$

Now $y^{(4)}(0)$:

$$y^{(4)} = 2y'' + y'' + xy''' \quad \Rightarrow \quad y^{(4)}(0) = 3y''(0) = 3 \cdot 2 = 6$$

The coefficients thus far are:

$$a_0 = 2, \quad a_1 = 1, \qquad a_2 = \frac{2}{2!} = 1, \qquad a_3 = \frac{2}{3!} = \frac{1}{3}, \qquad a_4 = \frac{6}{4!} = \frac{1}{4}$$

And the power series is:

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + h.o.t.$$

17. $y'' + xy + 2y = 0$, with $y(0) = 4$, $y'(0) = -1$.

SOLUTION:

From the initial conditions, $a_0 = 4$ and $a_1 = -1$. Now find $y''(0)$:

$$y'' = -xy' - 2y \quad \Rightarrow y''(0) = -0 \cdot y'(0) - 2y(0) \quad \Rightarrow \quad y''(0) = -8$$

Now $y'''(0)$:

$$y''' = -y' - xy'' - 2y' = -3y' - xy'' \quad \Rightarrow \quad y'''(0) = (-3)(-1) = 3$$

Now $y^{(4)}(0)$:

$$y^{(4)} = -3y'' - y'' - xy''' = -4y'' - xy''' \quad \Rightarrow \quad y^{(4)}(0) = -4y''(0) = (-4)(-8) = 32$$

Now the coefficients of our sum:

$$a_0 = 4, \qquad a_1 = -1, \qquad a_2 = \frac{-8}{2!} = -4, \qquad a_3 = \frac{3}{3!} = \frac{1}{2}, \qquad a_4 = \frac{32}{4!} = \frac{4}{3}$$

The power series is then:

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 + h.o.t.$$

18. $(1 - x)y'' + xy' - y = 0$ with $y(0) = -3$ and $y'(0) = 2$.

In this problem, we also want to determine the first five terms using the derivatives of y. It may be easiest to "differentiate in place" rather than solve for y'' first. That is, for $y''(0)$:

$$(1 - 0)y''(0) + 0 - y(0) = 0 \quad \Rightarrow \quad y''(0) = y(0) = -3$$

And now differentiate (use the product rule on $(1 - x)y''$):

$$-1y'' + (1 - x)y''' + y' + xy'' - y' = 0 \quad \Rightarrow \quad (1 - x)y''' + (x - 1)y'' = 0 \quad \Rightarrow$$

$$(1 - 0)y'''(0) - y''(0) = 0 \quad \Rightarrow \quad y'''(0) = y''(0) = -3$$

And one more time, using $(1 - x)y''' + (x - 1)y'' = 0$:

$$-1y''' + (1 - x)y^{(4)} + 1y'' + (x - 1)y''' = 0 \quad \Rightarrow$$

$$y^{(4)}(0) = -y''(0) + 2y'''(0) = 3 + 2(-3) = -3$$

Now, we can write the series (remember to divide by $n!$):

$$y(x) = -3 + 2x - \frac{3}{2!}x^2 - \frac{3}{3!}x^2 - \frac{3}{4!}x^4 + \cdots$$

For extra practice, we can use the technique in 15-18 to solve this one as well:

$$y'' + x^2y' + \sin(x)\, y = 0;$$