Review questions, Exam 3

1. What is the ansatz we use for y in: Chapter 6? Section 5.2?
2. Finish the definitions:
   - The Heaviside function, $u_c(t)$:
   - The Dirac $\delta-$function: $\delta(t-c)$ (Note: the Dirac function should be defined as a certain limit)
   - Define the convolution: $(f * g)(t)$
   - A function is of **exponential order** if:
3. Use the *definition* of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)
   $$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 6-t, & t \ge 2 \end{cases}$$

   (b)
   $$f(t) = \begin{cases} e^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

4. Check your answers to Problem 3 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.
5. Show that $f(t) = t^3$ is of exponential order. Repeat with $f(t) = \cos(t)$. (HINT: If needed, you may assume that $\ln(t) < t$ for $t > 0$).
6. Write the following functions in piecewise form (thus removing the Heaviside function):

   (a) $(t+2)u_2(t) + \sin(t)u_3(t) - (t+2)u_4(t)$

   (b) $\displaystyle\sum_{n=1}^{4} u_{n\pi}(t)\sin(t-n\pi)$

7. Determine the Laplace transform, using the table:

   (a) $t^2 e^{-9t}$

   (b) $e^{2t} - t^3 - \sin(5t)$

   (c) $t^2 y'(t)$ (in terms of $Y(s)$)

   (d) $e^{3t}\sin(4t)$

   (e) $e^t \delta(t-3)$

   (f) $t^2 u_4(t)$

8. Find the inverse Laplace transform, using the table:

   (a) $\dfrac{2s-1}{s^2-4s+6}$

   (b) $\dfrac{7}{(s+3)^3}$

   (c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$

   (d) $\dfrac{3s-1}{2s^2-8s+14}$

   (e) $\left(e^{-2s} - e^{-3s}\right)\dfrac{1}{s^2+s-6}$

9. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

   (b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$

   (c) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$

10. Solve the given initial value problems using Laplace transforms:

    (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.
    (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$
    (c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
    (d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$
    (e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

11. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

    (a) $4y'' + 4y' + 17y = g(t) \qquad y(0) = 0, y'(0) = 0$
    (b) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t)$, with $y(0) = 1$ and $y'(0) = -1$.

12. Short Answer:

    (a) $\displaystyle\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt =$ ____________________
    (b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:
    $$y'(t) = \delta(t-c), \qquad y(0) = 0$$
    (c) Use Laplace transforms to solve for $F(s)$, if
    $$f(t) + 2\int_0^t \cos(t-x) f(x)\, dx = e^{-t}$$
    (So only solve for the transform of $f(t)$, don't invert it back).
    (d) In order for the Laplace transform of f to exist, f must be ____________________
    (e) Is $x = 0$ an ordinary point for the differential equation: $xy'' + 3x^2y' + y = 4$?

13. Let $f(t) = t$ and $g(t) = u_2(t)$.

    (a) Use the Laplace transform to compute $f * g$.
    (b) Verify your answer by computing $f * g$ using the definition of the convolution.

14. If $a_0 = 1$, determine the coefficients a_n so that
    $$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$
    Try to identify the series represented by $\sum_{n=0}^{\infty} a_n x^n$.

15. Write the following as a single sum in the form $\sum_{k=2}^{\infty} c_k (x-1)^k$ (with perhaps a few terms in the front):
    $$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + x(x-2)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

16. Characterize ALL (continuous or not) solutions to
    $$y'' + 4y = u_1(t), \qquad y(0) = 1, y'(0) = 1$$
    (Hint: We could have solved this IVP without Laplace transforms. How?)

17. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to convert the following DE into a linear first order DE in $Y(s)$ (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

18. Find the recurrence relation between the coefficients for the power series solutions to the following:

   (a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.
   (b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$
   (c) $y'' - xy' - y = 0$, $x_0 = 1$

19. Exercises with the table:

   (a) Prove table entry 6 using entries 4 and 11.
   (b) Show that you can use table entry 15 to find the Laplace transform of $t^2\delta(t-3)$ (verify your answer using a property of the δ function).
   (c) Prove (using the definition of $\mathcal{L}$) table entries 9 and 10.
   (d) Prove (using the definition of $\mathcal{L}$) a formula (similar to 14) for $\mathcal{L}(y'''(t))$.

20. Find the first 5 terms of the power series solution to $e^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

21. Find the radius of convergence for all of the following, and find the interval of convergence for (b) and (d):

   (a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}x^n$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$