Solutions to Worksheet: Linear Operators

1. Let $R(f)$ be the operator defined by: $R(f) = f''(t) + 3t^2 f(t)$. Find $R(f)$ for each function below:

   (a) $f(t) = t^2$
   SOLUTION:
   Since $f'(t) = 2t$ and $f''(t) = 2$, we have:
   $$R(t^2) = 2 + 3t^2 \cdot t^2 = 2 + 3t^4$$

   (b) $f(t) = \sin(3t)$
   SOLUTION:
   Since $f'(t) = 3\cos(3t)$ and $f''(t) = -9\sin(3t)$, we substitute:
   $$R(\sin(3t)) = -9\sin(3t) + 3t^2 \sin(3t) = (3t^2 - 9)\sin(3t)$$

   (c) $f(t) = 2t - 5$
   SOLUTION:
   $$R(2t - 5) = 0 + 3t^2(t - 5) = 3t^3 - 15t^2$$

2. Let R be the operator defined in the previous problem. Show that R is a linear operator.
   SOLUTION:
   - $R(f + g) = (f + g)'' + 3t^2(f + g) = f'' + g'' + 3t^2 f + 3t^2 g = f'' + 3t^2 f + g'' + 3t^2 g = R(f) + R(g)$
   - $R(cf) = (cf)'' + 3t^2(cf) = cf'' + 3t^2 cf = c(f'' + 3t^2 f) = cR(f)$

3. Let $F(y) = y'' + y - 5$. Explain why F is not linear.
   SOLUTION:
   Best way is to show it- This F does not satisfy either part of the definition. For example,
   $$F(x + y) = (x + y)'' + (x + y) - 5 = x'' + x + y'' + y - 5 \neq (x'' + x - 5) + (y'' + y - 5) = F(x) + F(y)$$

4. Find the operator associated with the given differential equation, and classify it as linear or not linear:

   (a) $y' = ty^2 + \cos(t)$
   SOLUTION
   $$L(y) = y' - ty^2$$
   Not linear

   (b) $y'' = 4y' + 3y + \sin(t)$
   SOLUTION
   $$L(y) = y'' - 4y' - 3y$$
   Linear

   (c) $y' = \mathrm{e}^t y + 5$
   SOLUTION
   $$L(y) = y' - \mathrm{e}^t y$$
   Linear (in y)

(d) $y'' = -\cos(y) + \cos(t)$
SOLUTION

$$L(y) = y'' + \cos(y)$$

Not linear