HW to Replace 7.1: Systems of Differential Equations

1. For each second order differential equation below, write the corresponding system of first order equations. Lastly, write the system in matrix-vector form.

   (a) $y'' + 2y' - 3y = 0$

   (b) $y'' + 4y' + 4y = 0$

   (c) $y'' - 9y = 0$

   (d) $y'' - 2y' + 2y = 0$

2. For extra practice, go back and give the general solution to each second order DE (using methods from Chapter 3 of our textbook).

3. For each system of first order, convert to a corresponding second order differential equation.

   (a) $\begin{aligned} x' &= 3x - 2y \\ y' &= 2x - 2y \end{aligned}$

   (b) $\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$

   (c) $\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$

4. For extra practice, solve each of the second order DEs from the previous exercise using techniques from Chapter 3 of our text.

5. Consider the system below:

$$\begin{aligned} x' &= -3x + y \\ y' &= -2y \end{aligned}$$

   Solve this system by recognizing that we can solve for y directly, then substitute this into the DE for x and solve it as a first order linear DE.

6. Each system below is *nonlinear*. Solve each by first writing the system as dy/dx.

   (a) $\begin{aligned} x' &= y(1 + x^3) \\ y' &= x^2 \end{aligned}$

   (b) $\begin{aligned} x' &= 4 + y^3 \\ y' &= 4x - x^3 \end{aligned}$

   (c) $\begin{aligned} x' &= 2x^2y + 2x \\ y' &= -(2xy^2 + 2y) \end{aligned}$

7. Convert the third order (nonlinear) differential equation into a system of first order equations.

$$y''' - y'' + y'y = t^2$$

SOLUTIONS

1. For each second order differential equation below, write the corresponding system of first order equations.

   (a) $y'' + 2y' - 3y = 0$
   SOLUTION: Let $u = y, v = y'$. Then notice that $v' = y'' = 3y - 2y' = 3u - 2v$. The full system is then:

   $$\begin{aligned} u' &= v \\ v' &= 3u - 2v \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

   NOTE: Always write your variables in the same order. Since we started with u, v, then u' came first, then v', etc.

   (b) $y'' + 4y' + 4y = 0$
   SOLUTION: Let $u = y, v = y'$. Then $v' = y'' = -4y - 4y' = -4u - 4v$, and

   $$\begin{aligned} u' &= v \\ v' &= -4u - 4v \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

   (c) $y'' + 9y = 0$
   SOLUTION: Let $u = y, v = y'$. Then $v' = y'' = -9y = -9u$, and

   $$\begin{aligned} u' &= v \\ v' &= -9u \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

   (d) $y'' - 2y' + 2y = 0$
   SOLUTION: Let $u = y, v = y'$. Then $v' = y'' = 2y' - 2y$, and

   $$\begin{aligned} u' &= v \\ v' &= -2u + 2v \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

2. For extra practice, go back and give the general solution to each second order DE (using methods from Chapter 3 of our textbook).

   SOLUTIONS:

   (a) $y'' + 2y' - 3y = 0$
   The characteristic equation is $r^2 + 2r - 3 = 0$, or $(r + 3)(r - 1) = 0$, so $r = 1, -3$. Therefore, the general solution is

   $$y(t) = C_1 \mathrm{e}^{t} + C_2 \mathrm{e}^{-3t}$$

   (b) $y'' + 4y' + 4y = 0$
   The characteristic equation is $r^2 + 4r + 4 = 0$, or $(r + 2)^2 = 0$, so $r = -2, -2$. Therefore, the general solution is

   $$y(t) = \mathrm{e}^{-2t}(C_1 + C_2 t)$$

(c) $y'' + 9y = 0$

The characteristic equation is $r^2 + 9 = 0$, or $r = \pm 3i$,

$$y(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

(d) $y - 2y' + 2y = 0$

The characteristic equation is $r^2 - 2r + 2 = 0$, or $r = 1 \pm i$, so $r = 1, -3$. Therefore, the general solution is

$$y(t) = \mathrm{e}^t (C_1 \cos(t) + C_2 \sin(t))$$

3. For each system of first order, convert to a corresponding second order differential equation.

(a) $\begin{aligned} x' &= 3x - 2y \\ y' &= 2x - 2y \end{aligned}$

SOLUTION: Using the first equation to solve for y, $y = -(1/2)(x' - 3x)$. Putting this into the second equation:

$$\left(-\frac{x' - 3x}{2} \right)' = 2x - 2 \left(-\frac{x' - 3x}{2} \right)$$

Clean up by multiplying both sides by -2:

$$x'' - 3x' = -4x - 2(x' - 3x) \quad \Rightarrow \quad x'' - x' - 2x = 0$$

(b) $\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$

SOLUTION: Using the first equation to solve for y, $y = x' + 2x$. Putting this into the second equation:

$$(x' + 2x)' = x - 2\,(x' + 2x) \quad \Rightarrow \quad x'' + 4x' + 3x = 0$$

(c) $\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$

SOLUTION: Similar to the others, $y = x' - x$ from equation 1, so equation 2 becomes:

$$x'' - x' = 4x - 2(x' - x) \quad \Rightarrow \quad x'' + x' - 6x = 0$$

4. For extra practice, solve each of the second order DEs from the previous exercise using techniques from Chapter 3 of our text.

(a) $x(t) = C_1\mathrm{e}^{2t} + C_2\mathrm{e}^{-t}$ (b) $x(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-3t}$ (c) $x(t) = C_1\mathrm{e}^{2t} + C_2\mathrm{e}^{-3t}$

5. Consider the system below:

$$\begin{aligned} x' &= -3x + y \\ y' &= -2y \end{aligned}$$

Solve this system by recognizing that we can solve for y directly, then substitute this into the DE for x and solve it as a first order linear DE.

SOLUTION: Looking at $y' = -2y$, $y(t) = C_1\mathrm{e}^{-2t}$. Putting that into the equation for x', we get:

$$x' + 3x = C_1\mathrm{e}^{-2t}$$

The integrating factor is e^{3t}, so that

$$(x\mathrm{e}^{3t})' = C_1\mathrm{e}^{t} \quad \Rightarrow \quad x\mathrm{e}^{-3t} = C_1\mathrm{e}^{t} + C_2 \quad \Rightarrow \quad x(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{-3t}$$

As a parametric system, we have:

$$\begin{bmatrix} C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{-3t} \\ C_1\mathrm{e}^{-2t} \end{bmatrix}$$

We would note that the C_1 for y is the SAME as the C_1 for the x.

6. Each system below is *nonlinear*. Solve each by first writing the system as dy/dx.

(a) $\begin{aligned} x' &= y(1+x^3) \\ y' &= x^2 \end{aligned}$

SOLUTION: This becomes separable.

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \quad \Rightarrow \quad \int y\,dy = \int \frac{x^2\,dx}{1+x^3} \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}\ln(1+x^3) + C$$

At this point, we'll leave it since we won't know if we need the positive or negative root.

(b) $\begin{aligned} x' &= 4 + y^3 \\ y' &= 4x - x^3 \end{aligned}$

SOLUTION: This becomes separable:

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \quad \Rightarrow \quad \int 4 + y^3\,dy = \int 4x - x^3\,dx \quad \Rightarrow \quad 4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$$

We'll leave this in implicit form.

(c) $\begin{aligned} x' &= 2x^2y + 2x \\ y' &= -(2xy^2 + 2y) \end{aligned}$

SOLUTION: This becomes exact (don't cancel anything!)

$$\frac{dy}{dx} = -\frac{2xy^2 + 2y}{2x^2y + 2x} \quad \Rightarrow \quad (2xy^2 + 2y) + (2x^2y + 2x)\frac{dy}{dx} = 0$$

This is exact, with $M_y = N_x = 4xy + 2$. The solution is

$$x^2y^2 + 2xy = C$$

7. Convert the third order (nonlinear) differential equation into a system of first order equations.

$$y''' - y'' + y'y = t^2$$

SOLUTION: Let $x_1 = y, x_2 = y'$, and $x_3 = y''$. Then:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_1x_2 + x_3 + t^2 \end{aligned}$$