Summary: Chapter 6, 5.1-5.2

Here is a summary of the material in Chapter 6 and sections 5.1-5.2. For the material on the Laplace transform, a table will be provided, as usual.

6.1

- Know the definition of the Laplace transform. Be able to compute the corresponding limit (inc. l'Hospital's rule).
- When will the Laplace transform exist? If the function is piecewise continuous and of exponential order. Know these definitions; be able to determine if a function is of exponential order.
- Show that $\mathcal{L}$ is a linear operator.

6.2

- Prove (using the definition of $\mathcal{L}$):

  $$\mathcal{L}(f'(t)) = sF(s) - f(0) \qquad \mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

  where $F(s) = \mathcal{L}(f(t))$.
- Prove table entries: 1-3, 5-6, 9-10, 19.
- Use the table to invert the transform, assuming the inverse is linear (Here is where all the algebra comes in). When should we factor a quadratic in the denominator (versus complete the square)?

6.3-6.4

- Define the Heaviside function $u_c(t)$ and compute its Laplace transform (from the definition).
- The Heaviside function can be used as an "on-off" switch for the forcing function.
- Convert a piecewise defined function into an equivalent function using the Heaviside, and vice versa.
- Use the table entry: $\mathcal{L}(u_c(t)f(t-c)) = \mathrm{e}^{-cs}F(s)$. In particular, be able to take the transform of something like $u_3(t)\,t^2$.
- Use the table entry: $\mathcal{L}(\mathrm{e}^{ct}f(t)) = F(s-c)$. For example, use it to show Table Entry 11 from Table Entry 3, and Entries 9 and 10 from 5 and 6.
- The extra wrinkle introduced in 6.4 is to be able to solve the DE's when the forcing function (the function on the right hand side of the DE) uses summation notation.
- Be able to write out the solution as a piecewise defined function (without the Heaviside function).

6.5

- Define the Dirac $\delta-$function (or "unit impulse function"), and be able to compute its Laplace transform. Two important properties: $\displaystyle\int_{-\infty}^{\infty} \delta(t-c)\,dt = 1 \qquad \delta(t-c)f(t) = f(c)\delta(t-c)$
- Used to model a force of very short duration with finite strength (for example, a hammer strike). We saw that using the Dirac function is like imparting a velocity of $+1$ on the mass-spring system at $t = c$.
- Solve DEs that use the Dirac $\delta-$function. In particular, if the forcing function again uses summation notation (like in 6.4).

6.6

- Know the definition of the convolution, and be able to compute it directly for "simple" cases.
- Know "The Convolution Theorem": $\mathcal{L}^{-1}(F(s)G(s)) = f(t) * g(t)$.
- Use the Convolution Theorem to compute a convolution (using partial fractions).
- You do not need to know the *impulse response* or *transfer function* that are mentioned in the last example in the text.
- In the exercises, the new type of problem is the *integral equation*. Be able to solve these using the Laplace transform (like 23-28, part (a)).

5.1

- Review of power series and the ratio test for absolute convergence.
- Find the radius of convergence and the interval of convergence.
- Recall the template series: $\sum \frac{(-1)^n}{n} \qquad \sum \frac{1}{n} \qquad \sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}, |r| < 1$
- And the template Maclaurin series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1 \qquad \mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$

  And it is useful to recall the series for the sine (with odd powers) and cosine (with even powers) as well.
- Algebra: Be able to manipulate the index of summation (5.1, Examples 3-6)
- Know the underlying theorem for using series:

$$\sum_n c_n x^n = 0 \text{ for all } x \quad \Rightarrow \quad c_n = 0 \text{ for each } n$$

  This is the extension of a theorem we use all the time in solving partial fraction problems as well: If two polynomials are equal for all x, then their coefficients must be the same.

5.2

- Consider the model equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

  where P, Q, and R are polynomials. Then a point x_0 is called an **ordinary point** if $P(x_0) \neq 0$. Alternatively, if $P(x_0) = 0$, then the point is called a **singular point**. In Sections 5.2 and 5.3, we only find series solutions about ordinary points.
- Our ansatz for 5.1-5.3 is that y is analytic at x_0:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$$

- Be able to substitute the general series into a given DE, to find the **recurrence relation** for the coefficients.
- Be able to compute the series solution (up to a few terms) by directly computing the derivatives $(y''(0), y'''(0), y^{(4)}(0)$, etc.