Systems of ODEs (Replaces 7.1)

Topics today:

- What is a *system* of first order DEs?
- What is a solution to a system of ODEs?
  - What is a parametric equation? How to define the derivative/integral.
- What is a linear system (of ODEs)?
- Be able to convert a second order DE to a system of first order.
- Be able to convert a system of first order to a second order DE.
- Solve a system by first converting it to second order.
- Solve a system by first rewriting it as a single first order equation ("unparametrize" the curve).

Review- Parametric Equations

Recall from Calculus III that curves in the plane can be described by a parametric set of curves. For example, we know that a circle can be parametrized by:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

We also know how to differentiate and integrate parametric functions:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} \qquad \int_a^b \mathbf{x}(t)\,dt = \begin{bmatrix} \int_a^b x_1(t)\,dt \\ \int_a^b x_2(t)\,dt \end{bmatrix}$$

If we take the functions for the circle and differentiate them, what do we get?

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x' &= -y \\ y' &= x \end{aligned}$$

And this is how systems of differential equations comes naturally from parametrically defined functions. In fact, we see that the solution to the system is a parametric set of functions.

Definition of a system of DEs

A system of differential equations is any system of equations of the form:

$$\begin{aligned} x' &= f(x, y, t) \\ y' &= g(x, y, t) \end{aligned}$$

Here are some examples:

$$\begin{aligned} x' &= x^2 + xy + \sin(t) \\ y' &= 1 + y + \cos(t) \end{aligned} \qquad \begin{aligned} x_1' &= 3x_1 + 2x_2 \\ x_2' &= -x_1 + x_2 \end{aligned} \qquad \begin{aligned} x' &= x + 3y + 5 \\ y' &= \sin(xy) + t \end{aligned}$$

In this part of the course, we'll only look at the following- autonomous linear systems of differential equations:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

You might note that the system $x' = -y, y' = x$ was of this form.

Conversions: Linear System to 2d order

We'll show that often we can convert a linear system to a second order differential equation, which means we can solve the system using methods from Chapter 3. Here we'll do an example:

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

The idea is that we have two equations in two variables- We can algebraically manipulate them to make a single equation in one variable. For example, we can start with the first equation and solve for y:

$$x' = x + 3y \quad \Rightarrow \quad y = \frac{x'}{3} - \frac{x}{3}$$

Now substitute this expression in for y in the second equation:

$$y' = x - y \quad \Rightarrow \quad \left(\frac{x'}{3} - \frac{x}{3}\right)' = x - \left(\frac{x'}{3} - \frac{x}{3}\right) \quad \Rightarrow \quad \frac{x''}{3} - \frac{x'}{3} = x - \frac{x'}{3} + \frac{x}{3}$$

And simplifying algebraically- multiply through by 3 and bring everything over:

$$x'' - x' = 3x - x' + x \quad \Rightarrow \quad x'' - 4x = 0$$

Now we can solve this using Chapter 3, and get $x(t) = C_1 e^{2t} + C_2 e^{-2t}$. From this, we can now solve for y:

$$y = \frac{1}{3}(x' - x) = \frac{1}{3}\left((2C_1 e^{2t} - 2C_2 e^{-2t}) - (C_1 e^{2t} + C_2 e^{-2t})\right) = \frac{1}{3}\left(C_1 e^{2t} - 3C_2 e^{-2t}\right)$$

Alternate Solution:

Going back to the original system:

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

We could have used the second equation for the substitution, which would have avoided fractions:

$$x = y' + y \quad \Rightarrow \quad (y' + y)' = (y' + y) + 3y \quad \Rightarrow \quad y'' + y' = y' + 4y \quad \Rightarrow \quad y' - 4y = 0$$

Then $y(t) = C_3 e^{2t} + C_4 e^{-2t}$, and we can then solve for $x(t)$.

Example

Let's convert our first example and see if we get the right solution:

$$\begin{aligned} x' &= -y \\ y' &= x \end{aligned}$$

From the second equation, $x = y'$, so substituting that into the first equation, we get

$$x' = -y \quad \Rightarrow \quad y'' = -y \quad \Rightarrow \quad y'' + y = 0 \quad \Rightarrow \quad y(t) = C_1 \cos(t) + C_2 \sin(t)$$

And so $x = y' = -C_1 \sin(t) + C_2 \cos(t)$ for the general solution.

Conversions: 2d order to Systems

So far, we've converted system to 2d order equations. We can also do the reverse- Convert a second order DE into an equivalent system of equations. We do that by introducing a new set of variables.

Given: $ay'' + by' + cy = 0$. Then let $x_1 = y$, and $x_2 = y'$.

We now need to come up with equations for x_1' and x_2'. We see that $x_1' = y'$, and $y' = x_2$, so $x_1' = x_2$. Similarly, $x_2' = y''$, so we can substitute:

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad ax_2' + bx_2 + cx_1 = 0$$

Now the **system** of differential equations is given by:

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad \begin{array}{ll} x_1' & = x_2 \\ x_2' & = -\frac{c}{a}x_1 - \frac{b}{a}x_2 \end{array}$$

But don't memorize the formula- look at the technique we used, which works for any second order DE, not just linear homogeneous equations.

Example

Convert $y'' - yy' = \sin(t)$ into a system of equivalent first order DEs.

SOLUTION: Let $x_1 = y, x_2 = y'$. Now we make our usual substitutions to find that

$$\begin{array}{ll} x_1' & = x_2 \\ x_2' & = x_1x_2 + \sin(t) \end{array}$$

Example

See if you can verify the following conversion:

$$2y'' - 4y' + 6y = 0 \quad \Rightarrow \quad \begin{array}{ll} x_1' & = x_2 \\ x_2' & = -3x_1 + 2x_2 \end{array}$$

Now we can convert back and forth, we can solve many systems by converting them to second order, then solving using Chapter 3. Here's one more example before we discuss other techniques.

Example: Solve the system

$$\begin{array}{ll} x_1' & = 2x_1 + x_2 \\ x_2' & = -x_1 + 2x_2 \end{array}$$

SOLUTION: First convert to a second order DE (in either x_1 or x_2):

$$\begin{array}{ll} x_1' & = 2x_1 + x_2 \\ x_2' & = -x_1 + 2x_2 \end{array} \qquad x_2 = x_1' - 2x_1 \qquad \Rightarrow \quad x_1'' - 2x_1' = -x_1 + 2(x_1' - 2x_1)$$

so that $x_1'' - 4x_1' + 5x_1 = 0$, from which we have

$$r^2 - 4r + 5 = 0 \quad \Rightarrow \quad r = 2 \pm i$$

Therefore,

$$x_1(t) = \mathrm{e}^{2t}(C_1\cos(t) + C_2\sin(t))$$

and $x_2 = x_1' - 2x_1$, which, if we work it out, gives

$$x_2(t) = \mathrm{e}^{2t}(C_2\cos(t) - C_1\sin(t))$$

Therefore, using our methods from Chapter 3 (and Chapter 6), we can solve 2×2 systems of first order differential equations.

Solve a System by "Unparametrizing" it

This is really more of an observation than a formal technique. In Calc III, we learn that:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

so that we may be able to convert the system of equations (defining dx/dt and dy/dt) into dy/dx. For example, let's go back to our original system:

$$\begin{array}{ll} x' & = -y \\ y' & = x \end{array} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{y}$$

Now we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

which is separable, and

$$\int y\, dy = -\int x\, dx \quad \Rightarrow \quad \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

which is again the equation of a circle (but not in parametric form!).