Solutions - Homework (to replace 7.2)

1. What will the graph of $e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$ be (where t is any real number).

   SOLUTION: This is the set of all (positive) multiples of the vector, so it is a "ray" extending through the origin and through $(1, 2)$, and then outward.

2. Verify that $\mathbf{x}_1(t)$ below satisfies the DE below.

$$\mathbf{x}' = \begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}\mathbf{x}, \qquad \mathbf{x}_1(t) = e^{3t}\begin{bmatrix}1\\2\end{bmatrix}$$

   SOLUTION: Let $x_1(t) = e^{3t}$ and $x_2(t) = 2e^{3t}$. Then verify that:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$$

3. Each system below is *nonlinear*. Solve each by first writing the system as dy/dx.

   (a) $\begin{aligned} x' &= y(1+x^3) \\ y' &= x^2 \end{aligned} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x^2}{y(1+x^3)} \quad \Rightarrow \quad y\,dy = \dfrac{x^2}{1+x^3}\,dx$

   Now,

$$\frac{1}{2}y^2 = \frac{1}{2}\ln(1+x^3) + C$$

   (b) $\begin{aligned} x' &= 4+y^3 \\ y' &= 4x-x^3 \end{aligned} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{4x-x^3}{4+y^3} \quad \Rightarrow \quad (4+y^3)\,dy = 4x - x^3\,dx$

   Now,

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$$

   (c) $\begin{aligned} x' &= 2x^2y + 2x \\ y' &= -(2xy^2+2y) \end{aligned} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-(2xy^2+2y)}{2x^2y+2x}$

$$(2xy^2+2y) + (2x^2y+2x)\frac{dy}{dx} = 0$$

   This is exact, with $M_y = 4xy + 2$ and $N_x = 4xy + 2$.
   Antidifferentiating M with respect to x, we get:

$$x^2y^2 + 2xy$$

   If we differentiate that with respect to y, we get $2x^2y + 2x$, so that's our function. The solution is then:

$$x^2y^2 + 2xy = C$$

4. For each matrix below, compute the determinant $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$. Your determinant should be an expression in λ.

(a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

SOLUTION:

$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$

(b) $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$

SOLUTION:

$\begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = (3-\lambda)(-2-\lambda) + 4 = \lambda^2 - \lambda - 2$

(c) $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

SOLUTION:

$\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix} = (1-\lambda)(-4-\lambda) + 6 = \lambda^2 + 3\lambda + 2$

5. Show that if the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$ can be expressed as:

$$\lambda^2 - tr(A)\lambda + \det(A)$$

where $\text{tr}(A)$ is the trace of A and $\det(A)$ is the determinant of A.

SOLUTION: Expanding the determinant, you should get $\lambda^2 - (a+d)\lambda + (ad - bc)$.

6. Suppose $\lambda = 2$. Using the matrix $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$, solve the following system for v_1, v_2. Be sure to write your answer in vector form.

$$\begin{aligned} (a-\lambda)v_1 \quad & +bv_2 &= 0 \\ cv_1 \quad & +(d-\lambda)v_2 &= 0 \end{aligned}$$

SOLUTION: If $\lambda = 2$, the system of equations simplifies to a single line (the two lines are multiples of each other). We then write the line in parametric form:

$$\begin{aligned} v_1 - 2v_2 &= 0 \\ 2v_2 - 4v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

7. Suppose $\lambda = -1$. Using the matrix $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$, solve the following system for v_1, v_2. Be sure to write your answer in vector form.

$$\begin{aligned} (a-\lambda)v_1 \quad & +bv_2 &= 0 \\ cv_1 \quad & +(d-\lambda)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4v_1 - 2v_2 &= 0 \\ 2v_1 - v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$