General Review Questions

NOTE: You should also review past homework and exams.

1. Solve (use any method if not otherwise specified):
   (a) $(2x - 3x^2)\frac{dx}{dt} = t\cos(t)$
   (b) $y'' + 2y' + y = \sin(3x)$
   (c) $y'' - 3y' + 2y = \mathrm{e}^{2t}$
   (d) $x' = \sqrt{t}\mathrm{e}^{-t} - x$
   (e) $x' = 2 + 2t^2 + x + t^2 x$

2. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:
$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

3. The Wronskian of two functions is $W(t) = t^2 - 4$. Are they two linearly independent solutions to a second order linear differential equation? Why or why not?

4. Compute $\mathcal{L}(\cos(t))$ by using the definition of the Laplace transform.

5. Show that $y_1(t) = t$, $y_2(t) = t^2$ are linearly independent using the *definition* of linear independence. Compute the Wronskian of y_1 and y_2: Can they be linearly independent *solutions* to a second order linear differential equation?

6. Let $\mathbf{x}' = A\mathbf{x}$, where A is given below. Classify the origin (Poincaré Diagram), and give the general analytic solution.
   (a) $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$
   (b) $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$
   (c) $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$
   (d) $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$
   (e) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

7. Let $y''' - y' = t\mathrm{e}^{-t} + 2\cos(t)$. First, use our ansatz to find the characteristic equation for the third order homogeneous equation. Determine a suitable form for the particular solution, y_p using Undetermined Coefficients. Do not solve for the coeffs.

8. Write the differential equation associated with *Resonance* and *Beating*. Discuss under what conditions we can expect each type of behavior.

9. Suppose that we have a mass-spring system modelled by the differential equation
$$x'' + 2x' + x = 0, x(0) = 2, x'(0) = -3$$
Find the solution, and determine whether the mass ever crosses $x = 0$. If it does, determine the velocity at that instant. See if it crosses if the velocity is cut in half.

10. Let $y(x)$ be a power series solution to $(1-x)y'' + y = 0$, $x_0 = 0$. Find the recurrence relation, and write the solution to 6th order.

11. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0$, $x_0 = 1$. Find the recurrence relation and write the solution to 6th order.

12. Let $x' = \sin(y)$, $y' = \sin(x)$ Find all equilibria, and classify the stability.

13. Analyze how the origin changes classification with respect to α if:
$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ -\alpha & -2 \end{pmatrix}\mathbf{x}$$

14. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:
$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

15. Determine the Laplace transform:
   (a) $t^2\mathrm{e}^{-9t}$
   (b) $\mathrm{e}^{2t} - t^3 - \sin(5t)$
   (c) $u_5(t)(t-5)^4$
   (d) $\mathrm{e}^{3t}\sin(4t)$
   (e) $\mathrm{e}^t\delta(t-3)$
   (f) $t^2 u_4(t)$

16. Find the inverse Laplace transform:
   (a) $\dfrac{2s-1}{s^2 - 4s + 6}$
   (b) $\dfrac{7}{(s+3)^3}$
   (c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)}$
   (d) $\dfrac{3s-2}{(s-4)^2 - 3}$

17. Solve the given initial value problems using Laplace transforms:
   (a) $y'' + 2y' + 2y = 4t$, $y(0) = 0$, $y'(0) = -1$

(b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$
(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
(d) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$

18. Evaluate: $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt$

19. If $y'(t) = \delta(t - c)$, what is $y(t)$?

20. What was the *ansatz* we used to obtain the characteristic equation?

21. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

    (a) $y' - 0.5y = e^{2t} \qquad y(0) = 1$
    (b) $y'' + 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$
    (c) $y' = 1 + y^2$
    (d) $y' = \frac{1}{2}y(3 - y)$
    (e) $\sin(2x)\mathrm{dx} + \cos(3y)\mathrm{dy} = 0$
    (f) $y'' + 2y' + y = 2e^{-t}, \quad y(0) = 0, y'(0) = 1$
    (g) $y' = xy^2$
    (h) $2xy^2 + 2y + (2x^2 y + 2x)y' = 0$
    (i) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1.$

22. Suppose $y' = -ky(y - 1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

23. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

24. If $y(t)$ is a population at time t, what is the model for "exponential growth"? What is the model for growth with a "carrying capacity" in the environment? (Recall our Rabbit-Fox model).

25. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

26. For each nonlinear system, find and classify the equilibria:

    (a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 + 2y \\ 1 - 3x^2 \end{bmatrix}$

    (b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - y \\ x^2 - y^2 \end{bmatrix}$

    (c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + x^2 + y^2 \\ y(1 - x) \end{bmatrix}$

    (d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - y^2 \\ y - x^2 \end{bmatrix}$

27. Be sure to know the Existence and Uniqueness Theorem for $y' = f(t, y)$ (pg 66) and for linear equations, $y'' + p(t)y' + q(t)y = f(t)$ (pg 138).