Quiz Solutions

1. Solve $(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1$, where $y_1(t) = e^t$ and $y_2(t) = t$ are solutions to the homogeneous equation.

   We assume the particular solution is of the form:

   $$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

   where u_1' and u_2' solve the following:

   $$u_1'y_1 + u_2'y_2 = 0$$

   $$u_1'y_1' + u_2'y_2' = f(t)$$

   so that

   $$u_1' = \frac{-y_2 f(t)}{W(y_1, y_2)} \qquad u_2' = \frac{y_1 f(t)}{W(y_1, y_2)}$$

   We compute $W(e^t, t) = (1-t)e^t$, and $f(t)$ is found by putting the differential equation in standard form,

   $$y'' + \frac{t}{1-t}y - \frac{1}{1-t}y = -2(t-1)e^t$$

   Now,

   $$u_1' = \frac{-t(-2)(t-1)e^t}{(1-t)e^t} = -2t$$

   so we take $u_1(t) = -t^2$.

   Similarly,

   $$u_2' = \frac{e^t(-2)(t-1)e^t}{(1-t)e^t} = 2e^t$$

   so that $u_2 = 2e^t$

   Therefore, $y_p(t) = -t^2 e^t + (2e^t)t = -(t^2 - 2t)e^t$, and the full solution is:

   $$y(t) = C_1 e^t + C_2 t - e^t(t^2 - 2t)$$

2. Solve $x^2 y'' - 3xy' + 4y = x^2 \ln(x), \quad x > 0$, where $y_1(x) = x^2, y_2(x) = x^2 \ln(x)$ are the solutions to the homogeneous equation.

   We compute $W(x^2, x^2\ln(x)) = x^3$, and

   $$u_1' = \frac{-y_2 f(x)}{W(y_1, y_2)} = \frac{-x^2 \ln(x) \cdot \ln(x)}{x^3} = -\frac{(\ln(x))^2}{x}$$

   To get u_1, when integrating use the substitution $u = \ln(x)$ so that $du = (1/x)dx$ and

   $$u_1 = \int -\frac{(\ln(x))^2}{x}\, dx = -\int u^2\, du = -\frac{1}{3}u^3 = -\frac{1}{3}(\ln(x))^3$$

Similarly,

$$u_2 = \int \frac{x^2 \ln(x)}{x^3}\, dx = \frac{1}{2}(\ln(x))^2$$

so that

$$y_p(t) = -\frac{1}{3}(\ln(x))^3 \cdot x^2 + \frac{1}{2}(\ln(x))^2 \cdot x^2 \ln(x) = \frac{1}{6}x^2(\ln(x))^3$$

and the full solution is:

$$y(t) = C_1 x^2 + C_2 x^2 \ln(x) + \frac{1}{6}x^2(\ln(x))^3$$