Solutions to Review Questions, Ch 1-3.5

1. Definitions: Be sure to look these up if you're not sure of any of them.

2. The E&U Theorem states that if p, g are continuous on an open interval I containing t_0, then there exists a unique solution to the IVP: $y' + p(t)y = g(t)$, $y(t_0) = y_0$. This solution will be valid on I.

3. The E&U Theorem states that if $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on an open rectangle containing containing (t_0, y_0), then there exists a small interval about t_0, say $t_0 - h < t_0 < t_0 + h$ where there is a unique solution. In practice, to find the interval, we need to actually solve the differential equation.

4. True or False, and explain:

   (a) In general, this is true. We can draw a direction field for first order D.E.'s because the slope, y' depends only on location, $f(t, y)$. For a second order equation, we can think of y'' as depending on t, y and y'.

   (b) The solution to $y' = \sin(y)$ cannot be periodic: True. The solution to autonomous first order equations cannot oscillate because in the direction field along any horizontal line, all the slopes are the same.

   (c) False. In this case, $f(t, y) = y^{1/3}$, which is continuous everywhere, but $f_y = \frac{1}{3}y^{-2/3}$ is NOT continuous at $y = 0$. Therefore, we can have multiple solutions (we produced the multiple solutions in class).

   (d) If $y' = y^2$, then $-y^{-1} = t + C_1$ so that $y = \frac{-1}{t} + C_2$. This is False (the last step).
   $$y = \frac{-1}{t + C_1} \neq -\frac{1}{t} + C_2$$

   (e) True. On the interval $[0, 1]$, $|t| = t$, so that $3t$ is a multiple of $|t|$. On the interval $[-1, 1]$, these functions are not multiples of each other. We show that they are linearly independent by solving: $c_1 3t + c_2|t| = 0$. For one equation, use $t = 1$, and for the other, use $t = -1$, so that $c_1 = c_2 = 0$.

   (f) False. In the general case, if $W(f, g) = 0$ for all t, the functions may or may not be linearly independent.

   (g) This is slightly tricky- If f, g are solutions, then Abel's Theorem states that the Wronskian is either identically zero for all (valid) time, or never zero for all (valid) time. Therefore, the Wronskian may be zero at a point which is outside the interval on which the solutions to the D.E. are valid.

   (h) It's easy to see that $y = 0$ satisfies the differential equation. For the second,
   $$y' = -\sin(t) + \cos(t) \qquad y'' = -\cos(t) - \sin(t)$$

so that $y = \sin(t) + \cos(t)$ is also a solution. This does not violate the E&U Theorem- The theorem gives uniqueness for an IVP, not a general D.E.

(i) True, if p, q are continuous at t_0. In that case, we saw that we could find the fundamental set by solving two IVP's- y_1 solves the D.E. with $y(t_0) = 1, y'(t_0) = 0$. The function y_2 solves the D.E. with $y(t_0) = 0, y'(t_0) = 1$.

5. In the general case, since there is a t_0 so that the Wronskian is not zero, and assuming the two functions are differentiable, then the two functions are linearly independent.

6. **Typo in this problem:**

$$y(t) = \mathrm{e}^{-2t}\left(c_1\cos(3t) + c_2\sin(3t)\right)$$

This came from a second order linear D.E. with constant coefficients, where the two roots where:

$$r = -2 + 3i, -2 - 3i$$

This comes from a characteristic equation:

$$(r - (-2+3i))(r - (-2-3i)) = (r + 2 - 3i)(r + 2 + 3i) =$$

$$r^2 + 2r + 3ri + 2r + 4 + 6i - 3ri - 6i - 9i^2 = r^2 + 4r + 9$$

Thus, the D.E. was: $y'' + 4y' + 9y = 0$

7. Give all solutions:

(a) $y' = \sqrt{t}\mathrm{e}^{-t} - y$
This is linear, so put into standard form and use the integrating factor:

$$y' + y = t^{1/2}\mathrm{e}^{-t}, \qquad \text{I.F. } \mathrm{e}^{t}$$

$$(ty)' = t^{1/2} \Rightarrow (ty) = \frac{2}{3}t^{3/2} + C \Rightarrow y = \frac{2}{3}t^{1/2} + \frac{C}{t}$$

(b) $(2y - y^2)\frac{dy}{dt} = t\cos(t)$ This is separable (also exact). To integrate the right hand side, use integration by parts in a table:

$$y^2 - \frac{1}{3}y^3 = \cos(t) + t\sin(t) + C$$

(c) $y'' + 4y' + 5y = 0,\ y(0) = 1, y'(0) = 0$
Solve the characteristic equation, $r^2 + 4r + 5 = 0$ to get $r = -2 \pm i$. Then the general solution is:

$$y(t) = \mathrm{e}^{-2t}(c_1\cos(t) + c_2\sin(t))$$

Putting in the initial conditions gives a final answer of:

$$y(t) = \mathrm{e}^{-2t}(c_1\cos(t) + c_2\sin(t))$$

(d) $y' = \frac{1}{2}y(3-y)$ This is an autonomous D.E. We note that $y = 0$ and $y = 3$ are equilibrium solutions. If $y/neq0$ and $y/neq3$, then:

$$\frac{1}{y(3-y)}\,dy = \frac{1}{2}\,dt$$

Use partial fractions to integrate the left hand side of the equation to get:

$$\frac{1}{3}\ln|y| - \frac{1}{3}\ln|3-y| = \frac{1}{2}t + C$$

so that

$$\left(\frac{y}{3-y}\right) = A\mathrm{e}^{3/2t} \Rightarrow y = \frac{3}{1 + B\mathrm{e}^{-3/2t}}$$

(e) $y' = 2\cos(3x)$, $y(0) = 2$. Just integrate and solve:

$$y = \frac{2}{3}\sin(3x) + C \qquad 2 = 0 + C$$

$$y(t) = \frac{2}{3}\sin(3x) + 2$$

(f) $y' = 2 + 2t^2 + y + t^2y$ This is linear,

$$y' - (1+t^2)y = 2(1+t^2)$$

The integrating factor: $\mathrm{e}^{t+\frac{1}{3}t^3}$:

$$(\mathrm{e}^{t+\frac{1}{3}t^3}y)' = 2(1+t^2)\mathrm{e}^{t+\frac{1}{3}t^3}$$

Let $u = t + \frac{1}{3}t^3$ so that $du = (1+t^2)\,dt$, and

$$\mathrm{e}^{t+\frac{1}{3}t^3}y = 2\mathrm{e}^{t+\frac{1}{3}t^3} + C \qquad y(t) = 2 + C\mathrm{e}^{-t-\frac{1}{3}t^3}$$

(g) $y' = xy^2$, $y(0) = y_0$. This is separable. We also see that $y = 0$ is a possible solution, if $y_0 = 0$. Otherwise,

$$\frac{1}{y^2}\,dy = x\,dx \Rightarrow -\frac{1}{y} = \frac{1}{2}x^2 + C$$

At this stage, $C = -\frac{1}{y_0}$, so $y(t) = \dfrac{-2y_0}{y_0x^2 - 2}$

(h) $y' = 1 + y^2$ is an autonomous D.E.

$$\tan^{-1}(y) = t + C \Rightarrow y = \tan(t + C)$$

(i) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$ This is exact, which we check by letting $M(x,y) = 2xy^2 + 2y$, and $N(x,y) = 2x^2y + 2x$. Then:

$$M_y = 4xy + 2, \qquad N_x = 4xy + 2$$

Now $\Psi(x,y) = \int M(x,y)\,dx = x^2y^2 + 2xy + h(y)$, and we check that $\Psi_y = N(x,y)$:

$$\Psi_y(x,y) = 2x^2y + 2x + h'(y)$$

from which $h'(y) = 0$, so $h(y) = C$, which we take to be zero:

$$\Psi(x,y) = C_1 \Rightarrow x^2y^2 + 2xy = C_1$$

(j) $9y'' - 12y' + 4y = 0$, $y(0) = 0$, $y'(0) = -2$
In this case, we have one real root, $r = \frac{2}{3}$ so that the general solution is:

$$y(t) = \mathrm{e}^{\frac{2}{3}t}\,(C_1 + tC_2)$$

Solving for C_1, C_2, we get: $y(t) = -2\mathrm{e}^{\frac{2}{3}t}$

(k) $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -1$
In this case, we have one real root, $r = -1$ so that the general solution is:

$$y(t) = \mathrm{e}^{-t}\,(C_1 + tC_2)$$

Solving for C_1, C_2, we get: $y(t) = \mathrm{e}^{-t}$

(l) $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
In this case, we have two complex roots, $r = \pm 2i$ so that the general solution is:

$$y(t) = (C_1\cos(2t) + C_2\sin(2t))$$

Solving for C_1, C_2, we get: $y(t) = \cos(t) + \frac{1}{2}\sin(t)$

8. Let $y''+2y'+y = 0$, $y(0) = 2$, $y'(0) = -3$. Find the solution and determine if $y = 0$ for any time t. Same question if $y'(0) = -3/2$.

   The solution to the first IVP is $y(t) = \mathrm{e}^{-t}(2-t)$, which is zero for $t = 2$. The solution to the second IVP is $y(t) = \mathrm{e}^{-t}\left(2 + \frac{1}{2}t\right)$ which is zero for $t = -2$ (but is not zero in positive time).

9. The general solution is found by:

$$y(t) = \frac{3-\alpha}{5}\mathrm{e}^{-2t} + \frac{2+\alpha}{5}\mathrm{e}^{3t}$$

   Therefore, $\alpha = -2$ for $y \to 0$ as $t \to \infty$.

10. Determine the largest interval: The idea here is to use the fact that, for linear second order D.E.s: $y'' + p(t)y' + q(t)y = g(t)$, the Existence and Uniqueness Theorem requires p, q, g to be continuous. Writing the given ODE in standard form, we get:

$$y'' + \frac{3t}{t(t-4)}y' + \frac{4}{t(t-4)} = \frac{2}{t(t-4)}$$

    We will need to avoid $t = 0$ and $t = 4$, so that breaks the number line into three possibilities: $(4, \infty)$, $(0, 4)$ or $(-\infty, 0)$. Since we have $y(3)$ for an initial condition, we choose $(0, 4)$.

11. Solve $y' = ay + b$. We looked at several ways of solving this. First, let's check a:

   - If $a = 0$, $y' = b$, so $y(t) = bt + c$
   - If $a \neq 0$, the equilibrium is at $\frac{-b}{a}$, and the solution is $Ce^{at} - \frac{b}{a}$. If $a < 0$, the solutions will tend to equilibrium (the equilibrium is stable), and if $a > 0$, the equilibrium is unstable.

12. With
$$y(t) = \frac{3}{1 - 4e^{-3t}}$$
it is true that the limit is 3, but we have a vertical asymptote at $1 - 4e^{-3t} = 0$, which is positive. So starting at $t = 0$, the interval that is valid is approximately $(0, 0.462)$

13. For the tank mixing, take (Rate In)-(Rate Out): Let $Q(t)$ be the pounds of salt at time t. Then
$$\frac{dQ}{dt} = 4\frac{\text{gal}}{\text{min}} \cdot \frac{1}{2}\frac{\text{pound}}{\text{gal}} - 4\frac{\text{gal}}{\text{min}} \cdot \frac{Q(t)}{100}\frac{\text{pounds}}{\text{gal}}, \quad Q(0) = 100$$
Solve for Q, and get:
$$Q(t) = 50 + 50e^{-\frac{1}{25}t}$$
The concentration will be 0.7 when $Q(t)$ is 70,
$$70 = 50 + 50e^{-\frac{1}{25}t}$$
so that $t = -25\ln(2/5) \approx 22.91$ minutes.

14. We'll do this one in class.