The Important Stuff from 3.2 and 3.3

The last handout was a summary of how the theory of linear algebra and the theory of linear differential equations are intricately interwoven. While this is interesting and intriguing (and is continued in Chapter 4), we only had a short time to talk about it.

So, here is a brief summary of the most important definitions and theorems about Second Order, Linear, Homogeneous Differential Equations (SOLHDE, for you engineers!).

1. The Existence and Uniqueness Theorem.

   Let $y'' + p(t)y' + q(t)y = f(t)$, $y(t_0) = y_0$. Then if p, q, f are all continuous for $t \in (a, b)$, (and of course, $t_0 \in (a, b)$), then there exists a unique solution to the differential equation, and this solution persists for all $t \in (a, b)$.

2. Definitions:

   (a) Linear Independence: A set of functions, $\{y_1(t), \ldots, y_k(t)\}$ is said to be linearly independent on the interval $[a, b]$ iff the only solution to:
   $$c_1y_1(t) + \ldots + c_ky_k(t) = 0$$
   (for all $t \in [a, b]$) is: $c_1 = c_2 = \ldots = c_k = 0$

   (b) The Wronskian of y_1 and y_2 is:
   $$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$
   Note that this is a function of t.

   (c) The Fundamental Set of Solutions: y_1 and y_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$ if the solution to an aribtrary initial value problem can be written as a linear combination of y_1 and y_2. Thus, ALL solutions are of the form:
   $$c_1y_1 + c_2y_2$$
   if y_1 and y_2 form a fundamental set.

3. Theorem (Linear Independence and the Wronskian, in General)

   If f, g are differentiable on an open interval I and if $W(f, g)(t_0) \neq 0$ for some t_0 in I, then f and g are linearly independent on I.

   However, it is possible in this general case, that the Wronskian is zero for all t in I, and the functions f and g are linearly independent. This is not true in the special case that f, g are two solutions to the second order linear, homogeneous differential equation:

4. Abel's Theorem (Linear Independence and the Wronskian, for Solutions to a Linear Homogeneous Differential Equation).

   Let $y'' + p(t)y' + q(t)y = 0$, and let I be an open interval on which p and q are continuous. Let y_1 and y_2 be solutions. Then the Wronskian of y_1 and y_2 is either identically zero on I, (in which case y_1, y_2 are linearly dependent) or it is never zero on I (in which case y_1, y_2 are linearly independent).

   This is from the fact that we can compute the Wronskian *without knowing* y_1, y_2:
   $$W(y_1, y_2)(t) = c\mathrm{e}^{-\int p(t)\,dt}$$