Converting a trig sum to a single periodic function

The conversion formula we use in ODEs comes from the trig identity:

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

If we to determine R, δ so that:

$$C_1 \cos(\omega t) + C_2 \sin(\omega t) = R\cos(\omega t - \delta)$$

we can say that:

$$R\cos(\omega t - \delta) = R\left(\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)\right)$$

$$= (R\cos(\delta))\cos(\omega t) + (R\sin(\delta))\sin(\omega t)$$

Therefore,

$$C_1 = R\cos(\delta) \text{ and } C_2 = R\sin(\delta)$$

From these equations, we can get formulas for R, δ in terms of C_1, C_2:

$$C_1^2 + C_2^2 = R^2\cos^2(\delta) + R^2\sin^2(\delta) = R^2$$

so that $R = \sqrt{C_1^2 + C_2^2}$

Furthermore,

$$\frac{C_2}{C_1} = \frac{R\sin(\delta)}{R\cos(\delta)} = \tan(\delta)$$

so that $\tan^{-1}\left(\frac{C_2}{C_1}\right) = \delta$ if $-\frac{\pi}{2} < \delta < \frac{\pi}{2}$ (recall that the domain of the tangent needs a restriction so that it is invertible).

We need to be a bit careful in this computation. With $C_1 = R\cos(\delta)$ and $C_2 = R\sin(\delta)$, we can visualize C_1 and C_2 as coordinates on the circle of radius R, where C_1 is the "x-"coordinate and C_2 is the "y-"coordinate. If $x > 0$, then we can use the angle that the calculator provides:

If $C_1 > 0$, then $\tan^{-1}(C_2/C_1) = \delta$.

If $x < 0$, then we should be in Quadrants II or III of the unit circle, but the calculator will produce angles in Quadrants I and IV. Therefore:

If $C_1 < 0$, add π to the number provided by the calculator, $\delta = \tan^{-1}(C_2/C_1) + \pi$. Some calculators provide a four-quadrant inverse. If yours does this, the computer command will be something like: `arctan(y,x)` where you have to input two numbers instead of the fraction.

Examples: Be sure to try these yourself!

1. Rewrite $-\frac{1}{2}\cos(t) + \frac{\sqrt{3}}{2}\sin(t)$ as $R\cos(\omega t - \delta)$.

   In this case, $R = 1$ and $\delta = \tan^{-1}(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

Therefore,

$$-\frac{1}{2}\cos(t) + \frac{\sqrt{3}}{2}\sin(t) = \cos\left(t - \frac{2\pi}{3}\right)$$

NOTE: If you don't want to remember to add π in some cases, we can ALWAYS make $C_1 > 0$. For example, an alternative solution to this problem would be:

$$-\left(\frac{1}{2}\cos(t) - \frac{\sqrt{3}}{2}\sin(t)\right) = -R\cos(t - \delta)$$

so that $R = 1$ and $\delta = \frac{-\pi}{3}$:

$$-\left(\frac{1}{2}\cos(t) - \frac{\sqrt{3}}{2}\sin(t)\right) = -\cos\left(t + \frac{\pi}{3}\right)$$

2. Solve $y'' + y' + 2y = 0$, $y(0) = \frac{1}{2}, y'(0) = -1$, and write the solution as $Ae^{\alpha t}\cos(\omega t - \delta)$.

   The solutions to the characteristic equation, $r^2 + r + 2 = 0$ are

$$r = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

   so that the general solution is:

$$y(t) = e^{-\frac{t}{2}}\left(C_1\cos\left(\frac{\sqrt{7}}{2}t\right) + C_2\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

   Solving for C_1, C_2, we get:

$$y(t) = e^{-\frac{t}{2}}\left(\frac{1}{2}\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{3}{2\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

   so that $R = \frac{2}{\sqrt{7}}$ and $\delta = \tan^{-1}\left(\frac{-3}{\sqrt{7}}\right) \approx -0.848$

   Therefore,

$$y(t) = \frac{2}{\sqrt{7}}e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{7}}{2} + 0.848\right)$$