Review Problems, Exam 2, ODE

1. Solve using the method of undetermined coefficients:

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \; y'(0) = 0$$

SOLUTION: We break this into two parts, since $u_2(t) = 0$ if $t < 2$ and $u_2(t) = 1$ for $t \geq 2$. When $t = 0$, the solution is $y(t) = 0$. At time $t = 2$, we want y, y' to be continuous, so we have the same initial conditions, $y(2) = 0$, $y'(2) = 0$. Now we're solving:

$$y'' + 3y' + 2y = 1, \quad y(2) = 0 \; y'(2) = 0$$

The homogeneous part: $r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \Rightarrow y_h(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-2t}$. Now we guess that the particular solution has the form: $y_p(t) = A$, and we find that $A = \frac{1}{2}$. The general solution is therefore $y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-2t} + \frac{1}{2}$, and we solve for C_1, C_2:

$$y(2) = 0 = C_1\mathrm{e}^{-2} + C_2\mathrm{e}^{-4} + \frac{1}{2}$$

$$y'(2) = 0 = -C_1\mathrm{e}^{-2} - 2C_2\mathrm{e}^{-4}$$

From which we have $C_1 = -\mathrm{e}^2$ and $C_2 = \frac{1}{2}\mathrm{e}^4$. Putting this back in,

$$y(t) = -\mathrm{e}^2\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^4\mathrm{e}^{-2t} + \frac{1}{2} = -\mathrm{e}^{-(t-2)} + \frac{1}{2}\mathrm{e}^{-2(t-2)} + \frac{1}{2} \qquad t \geq 2$$

2. Solve the previous problem using Laplace transforms. To compare solutions, you should not use hyperbolic sine/cosine (so do a full partial fraction decomposition where appropriate).

$$(s^2 + 3s + 2)Y(s) = \frac{\mathrm{e}^{-2s}}{s} \Rightarrow Y(s) = \mathrm{e}^{-2s}\frac{1}{s(s+1)(s+2)} = \mathrm{e}^{-2s}H(s)$$

Now the solution will be $u_2(t)h(t-2)$, so we solve for $h(t)$:

$$H(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}$$

by partial fractions. Now,

$$h(t) = \frac{1}{2} - \mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{-2t}$$

and the full solution is:

$$y(t) = u_2(t)h(t-2) = u_2(t)\left(-\mathrm{e}^{-(t-2)} + \frac{1}{2}\mathrm{e}^{-2(t-2)} + \frac{1}{2}\right)$$

3. Let $y'' + ky' + \omega^2 y = A\cos(\omega_0 t)$. What is the physical interpretation of k if this models a pendulum or a spring?

   The constant k represented the coefficient of friction.

   Under what conditions on k, ω will the solution to the differential equation exhibit *beating*?

   For beating, $k = 0$ and ω is close to ω_0 (See p. 201)

   Resonance? For resonance, $k = 0$ and $\omega = \omega_0$ (See p. 202).

4. Solve using the method of Variation of Parameters, then check your answer using the method of undetermined coefficients:

$$3y'' - 3y' - 6y = 6\mathrm{e}^{-t}$$

   Did you remember to put the differential equation into standard form FIRST?

$$y'' - y' - 2y = 2\mathrm{e}^{-t}$$

   Now get the homogeneous part of the solution: $y_1 = \mathrm{e}^{2t}$, $y_2(t) = \mathrm{e}^{t}$.

   We'll need the Wronskian: $-3\mathrm{e}^{t}$.

   Now

$$u_1' = -\frac{y_2 g(t)}{W} = -\frac{\mathrm{e}^t \cdot 2\mathrm{e}^{-t}}{-3\mathrm{e}^t} = \frac{2}{3}\mathrm{e}^{-3t} \Rightarrow u_1(t) = -\frac{2}{9}\mathrm{e}^{-3t}$$

$$u_2' = \frac{y_1 g(t)}{W} = \frac{\mathrm{e}^{2t} \cdot 2\mathrm{e}^{-t}}{-3\mathrm{e}^t} = -\frac{2}{3} \Rightarrow u_2(t) = -\frac{2}{3}t$$

   Putting the particular solution together,

$$y_p(t) = u_1(t)y_1 + u_2(t)y_2 = -\frac{2}{9}\mathrm{e}^{-3t}\mathrm{e}^{2t} - \frac{2}{3}t\mathrm{e}^{-t} = \frac{2}{3}t\mathrm{e}^{-t}$$

   (Note: We can discard the first part of the solution since it is part of the homogeneous solution) The general solution is:

$$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{2t} - \frac{2}{3}t\mathrm{e}^{-t}$$

   We'll leave the double-checking using undetermined coefficients to you- remember to multiply your initial ansatz by t.

5. The Variation of Parameters method is useful is describing the solution to a differential equation with a generic forcing function $g(t)$. Give the particular solution using this method to:

$$y'' - 5y' + 6y = g(t)$$

   and rewrite your answer as a single integral.

The homogeneous part of the solution is $y_1 = e^{2t}$, $y_2(t) = e^{3t}$. The Wronskian is e^{5t}.

$$u_1' = -\frac{e^{3t} \cdot g(t)}{e^{5t}} \Rightarrow u_1(t) = \int g(t) e^{-2t}\, dt$$

$$u_2' = \frac{e^{2t} \cdot g(t)}{e^{5t}} \Rightarrow u_2(t) = \int g(t) e^{-3t}\, dt$$

Now,

$$y_p(t) = e^{2t} \int g(t) e^{-2t}\, dt + e^{3t} \int g(t) e^{-3t}\, dt$$

In order to write this as a single integral, we need to use a different dummy variable in the integration, for example, use ds instead of dt:

$$y_p(t) = e^{2t} \int g(s) e^{-2s}\, ds + e^{3t} \int g(s) e^{-3s}\, dt = \int g(s) \left(e^{-2(t-s)} + e^{-3(t-s)} \right)\, ds$$

6. Suppose that the solution to a second order linear homogeneous differential equation with constant coefficients is:

$$y_h(t) = C_1 e^{-2t} + c_2 t e^{-2t}$$

If the forcing function is given to by the expression to the left of the table, write in the ansatz for the particular solution in the column to the right:

Forcing Function	Give final ansatz for $y_p(t)$
$3t^2 - 1$	$At^2 + Bt + C$
$5e^t + 2e^{-2t}$	$Ae^t + Bte^{-2t}$
$t \sin(3t)$	$(At + B)\sin(3t) + (Ct + D)\cos(3t)$

7. True or False, and explain: The solution to every second order differential equation can be written as: $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is the solution to the homogeneous equation, and $y_p(t)$ is the particular solution.

   False. This can only be done when the differential equation is *linear*.

8. Let $y'' - xy' - y = 0$. Find the recurrence relation between the coefficients of the power series solution based at (i) $x_0 = 0$ and (ii) $x_0 = 1$.

   First, find a relationship between derivatives:

$$y^{(k+2)} = xy^{(k+1)} + (k+1)y^{(k)}$$

so at $x_0 = 0$, $y^{(k+2)}(0) = (k+1)y^{(k)}$. Substitute using $C_k = \frac{y^{(k)}(x_0)}{k!}$

$$C_{k+2} = \frac{1}{k+2} C_k$$

and at $x_0 = 1$,

$$C_{k+2} = \frac{1}{k+2}\left(C_{k+1} + C_k\right)$$

9. Find the first five coefficients of the power series solution to $(4 - x^2)y'' + 2y = 0$ if $y(0) = 0$ and $y'(0) = 1$. Write the solution up to 5th order using your answer.

   You should find that $y(x) = x - \frac{1}{12}x^3 + \frac{1}{240}x^5 + \dots$

10. (a) Rewrite the following as a single sum whose generic term involves x^n:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x_n$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + a_n\right) x^n$$

   (b) If this sum must be zero for all x, what relation between the a_n must hold? HINT: If $\sum c_n x^n = 0$ for all x, then $c_n = 0$ for all n.

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n$$

11. If $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, substitute this into the differential equation in Problem 8. Rewrite the expression as a single sum equal to zero, and solve for the recurrence relation at $x_0 = 0$. You'll use the technique from Problem 10.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

   Rewrite to get:

$$\sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} - n a_n - a_n)x^n = 0$$

   so that

$$a_{n+2} = \frac{1}{n+2} a_n$$