Practice Problems: Systems of Equations

1. Convert the following systems of equations into equivalent matrix-vector equations:
   (a) $3x - 5y = 2,\ -y + x = 0$
   (b) $4x - y + z = 1,\ -x + y = 1,\ 2x - z = 4$ (Note there are three equations in three unknowns).
   (c) $2x + y = -1,\ -x + 5y = 1$

2. Convert the following matrix-vector equations into equivalent systems of equations:
   (a) $$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
   (b) $$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
   (c) $$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3. Convert the following n^{th} order equations into equivalent systems of first order equations. If linear, further convert the system into matrix-vector form.
   (a) $y'' + 3y' - 5y = 0$
   (b) $y''' - y'\sin(t) + y^2 = \cos(t)$
   (c) $y^{(iv)} - y''' = 0$
   (d) $y'' + 2y' + y = 0$

4. Convert the following systems of equations into equivalent second order differential equations.
   (a) $$\begin{aligned} x_1' &= 3x_1 - 2x_2 \\ x_2' &= 2x_1 - 2x_2 \end{aligned}$$
   (b) $$\begin{aligned} x_1' &= 1.25x_1 + 0.75x_2 \\ x_2' &= 0.75x_1 + 1.25x_2 \end{aligned}$$
   (c) $$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

5. Use the "inverse" of the matrix to solve the following system of equations. If the determinant is zero, determine if there is no solution or an infinite number of solutions- and if there are an infinite number, what relationship must hold?
   (a) $$\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
   (b) $$\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
   (c) $$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
   (d) $$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
   (e) $$\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

6. Solve the following systems by first re-writing as an equivalent second order d.e.:
   (a) $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   with $x_1(0) = 1,\ x_2(0) = 5$
   (b) $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   with $x_1(0) = 1,\ x_2(0) = -1$
   (c) $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   with $x_1(0) = 1,\ x_2(0) = 0$