Laplace Transforms Review Solutions

1. Compute transforms from the definition:

   (a) $\int_0^2 3e^{-st}\,dt + \int_2^\infty (6-t)e^{-st}\,dt = \frac{3}{s} + \frac{9}{s}e^{-2s} - \frac{3}{s^2}e^{-2s}$

   (b) $\int_0^5 e^{-t}e^{-st}\,dt - \int_5^\infty e^{-st}\,dt = \frac{1-e^{-5(s+1)}}{s+1} - \frac{e^{-5s}}{s}$

2. Check your answers using Heaviside and table.

3. Compute transforms (using the table)

   (a) $\frac{2}{(s+9)^3}$

   (b) $\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$

   (c) $e^{-5s}\frac{4!}{s^5}$

   (d) $\frac{4}{(s-3)^2+16}$

   (e) Use Table #14, with $f(t) = \delta(t-3)$, so $e^{-3(s-1)}$.

   (f) Let $f(t-4) = t^2$, so $f(t) = (t+4)^2$: $e^{-4s}\left(\frac{2}{s^3} + \frac{8}{2^2} + \frac{16}{s}\right)$

4. Invert the transforms:

   (a) First rewrite as $\frac{2s-1}{(s-2)^2+2}$, so $2e^{\sqrt{2}t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}e^{2t}\sin(\sqrt{2}t)$

   (b) Via partial fractions: $e^{-t} - 3e^{-3t} + 3e^{2t}$

   (c) $\frac{7}{2}e^{-3t}t^2$

   (d) $u_2(t)[2e^{-2(t-2)} + 2e^{t-2}]$

   (e) Rewrite to get: $3\frac{(s-4)}{(s-4)^2+3} + \frac{10}{\sqrt{3}}\frac{\sqrt{3}}{(s-4)^2+3}$, so we get: $3e^{4t}\cos(\sqrt{3}t) + \frac{10}{\sqrt{3}}e^{4t}\sin(\sqrt{3}t)$

   (f) Write as $e^{-2s}H(s)$.

   $$H(s) = \frac{1}{s^2+2s-2} = \frac{1}{(s+1)^2-3} = F(s+1)$$

   where $F(s) = 1/(s^2-3)$. This gives $f(t) = 1/\sqrt{3}\ \sinh(\sqrt{3}t)$. Final answer:

   $$u_2(t)\cdot\frac{1}{\sqrt{3}}e^{-t}\sinh(\sqrt{3}t)$$

5. Solve the diff. eqn:

   (a) $e^{2t} - e^{5t}$

   (b) $-3e^{-3t} + te^{-3t}$

   (c) $\frac{1}{2}t^2 - t + \frac{5}{2} - \frac{5}{2}e^{-t}(\cos(t) + \sin(t))$

   (d) $\frac{10}{13}e^{2t} - \frac{23}{13}\cos(3t) + \frac{15}{13}\sin(3t)$

   (e) Let $h(t) = \frac{-1}{3} + \frac{1}{12}e^{3t} + \frac{1}{4}e^{-t}$ Then the solution is: $y(t) = u_1(t)h(t-1) - \frac{1}{4}e^{3t} + \frac{1}{4}e^{-t}$.

(f) $e^{2t}(2t-6)+e^{t}(t^2+4t+6)$

(g) $\frac{1}{2}\sin(2t)+\frac{1}{2}u_{\frac{\pi}{2}}(t)\sin(2(t-\frac{\pi}{2}))$

(h) $y(t)=\sum_{k=1}^{\infty}u_{2\pi k}(t)\sin(t)$ Note that this is:

$$y(t)=\begin{cases}\sin(t), & 0\le t<2\pi\\ 2\sin(t), & 2\pi\le t<4\pi\\ 3\sin(t), & 4\pi\le t<6\pi\\ \vdots & \vdots\end{cases}$$

6. Miscellaneous:

(a) Evaluate: $\displaystyle\int_0^{\infty}\sin(3t)\delta(t-\frac{\pi}{2})\,dt$

We know that $\displaystyle\int_{-\infty}^{\infty}\delta(t-c)f(t)\,dt=f(c)$. Therefore, for $c>0$,

$$\int_0^{\infty}\sin(3t)\delta(t-\pi/2)\,dt=\sin\left(\frac{3\pi}{2}\right)=1$$

(b) **Delete this problem, it's from 6.6**

(c) Use the table to find an expression for $\mathcal{L}(ty')$. Use this to solve:

$$y''+3ty'-6y=1,\quad y(0)=0,\quad y'(0)=0$$

From the table, $\mathcal{L}(ty')=-Y(s)-sY'(s)$. Take the Laplace transform, and:

$$s^2Y(S)+3(-Y-sY')-6Y=\frac{1}{s}\Rightarrow Y'+\left(\frac{s^2-9}{-3s}\right)Y=-\frac{1}{3s}$$

Use the method of the integrating factor:

$$\int\frac{s^2-9}{-3s}\,ds=\int-\frac{1}{3}s+\frac{3}{s}\,ds=-\frac{1}{6}s^2+3\ln(s)$$

Now the integrating factor is $e^{(-1/6)s^2+\ln(s^3)}=s^3e^{(-1/6)s^2}$:

$$(s^3e^{(-1/6)s^2}Y)'=-\frac{1}{3s}s^3e^{(-1/6)s^2}=-\frac{1}{3}se^{(-1/6)s^2}$$

so that

$$s^3e^{(-1/6)s^2}Y=e^{(-1/6)s^2}\;\Rightarrow\;Y(s)=\frac{1}{s^3}$$

Therefore, $y(t)=\frac{1}{2}t^2$.

7. Characterize all solutions: We solve by our old method of getting the homogeneous and particular solutions.

$$y(t)=\begin{cases}\cos(2t)+\frac{1}{2}\sin(2t), & 0\le t<1\\ c_1\cos(2t)+c_2\sin(2t)+\frac{1}{4}, t\ge 1\end{cases}$$

8. Define the delta function:

$$\delta(t-c) = \lim_{h \to 0} d_h(t-c)$$

where

$$d_h(t-c) = \begin{cases} \frac{1}{2h}, & c-h < t < c+h \\ 0, & \text{otherwise} \end{cases}$$

9. If $y'(t) = \delta(t-c)$, what is $y(t)$?

$$sY = \mathrm{e}^{-cs} \rightarrow Y = \frac{\mathrm{e}^{-cs}}{s} \rightarrow y(t) = u_c(t)$$

10. $\int_0^\infty \mathrm{e}^{-st} g(t+c)\, dt = \int_c^\infty \mathrm{e}^{-s(u-c)} g(u)\, du = \mathrm{e}^{cs} \int_c^\infty \mathrm{e}^{-su} g(u)\, du = \mathrm{e}^{cs} \left(G(s) - \int_0^c \mathrm{e}^{-st} g(t)\, dt\right)$