Summary Sheet: Analyze Non-Nonlinear Systems

1. In calculus, we learned how to linearize $y = f(x)$ at $x = a$:

$$y = f(x) \text{ is replaced by } y - f(a) = f'(a)(x - a)$$

   which is the equation of its tangent line at $x = a$.

   PROBLEM: Recall that, $f(x) = mx + b$ is not a linear function in the formal sense (a linear function satisfies the properties that $f(a + b) = f(a) + f(b)$, and $f(cx) = cf(x)$). We can solve this problem by introducing **local coordinates** (u, v), where $u = x - a$ and $v = y - f(a)$. Then,

$$y - f(a) = f'(a)(x - a) \text{ becomes the linear function } v = f'(a)u$$

2. Though we won't be working with these functions, for completeness we include them. In Calculus III, to linearize $f(x, y) = x^2 - 2xy - \sin(y)$ at $x = a, y = b$ means to replace f by the equation of the tangent plane:

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

   EXAMPLE: Linearize $z = x^2 - 2xy - \sin(y)$ at $x = 1, y = 0$:

$$f(1, 0) = 1, \quad \nabla f = (f_x, f_y) = (2x - y, -2x - \cos(y)), \quad \nabla f(1, 0) = (2, -3)$$

   so that the linearization is:

$$z - 1 = 2(x - 1) - 3(y - 0)$$

3. Linearization for Systems:

   First, we define the Jacobian Matrix:

$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ \\ f_2(x_1, x_2) \end{bmatrix} \quad \Rightarrow \quad Df(x_1, x_2) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}$$

   Then the linearization of f at $\mathbf{a}$ is:

$$\boxed{y - f(\mathbf{a}) = Df(\mathbf{a})(\mathbf{x} - \mathbf{a})}$$

   or in local coordinates, $\mathbf{u} = \mathbf{x} - \mathbf{a}$, $\mathbf{v} = \mathbf{y} - f(\mathbf{a})$,

$$\boxed{\mathbf{v} = Df(\mathbf{a})\mathbf{u}}$$

   Thus, we can study some nonlinear functions *locally* via linearization.

   EXAMPLE: Linearize the following function at $(0, \frac{\pi}{2})$. Write the answer in global, then local, coordinates.

$$f(x, y) = \begin{bmatrix} x + \sin(y) \\ xy_+ x^2 \end{bmatrix}$$

   SOLUTION:

$$Df = \begin{bmatrix} 1 & \cos(y) \\ y + 2x & x \end{bmatrix}, \text{ so } Df(0, \frac{\pi}{2}) = \begin{bmatrix} 0 & 1 \\ \frac{\pi}{2} & 0 \end{bmatrix}$$

   which gives us the following approximations:

$$\boldsymbol{z} - \begin{bmatrix} 1 \\ \frac{\pi}{2} + 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\pi}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ In local coordinates, } \boldsymbol{v} = \begin{bmatrix} 0 & 1 \\ \frac{\pi}{2} & 0 \end{bmatrix} \boldsymbol{u}$$

EXAMPLES

1. Linearize f at the given value: $f(x_1,x_2) = \begin{bmatrix} x_1 - 3x_1x_2 \\ x_2 + x_1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

   SOLUTION: $f(2,1) = [-4,3]^T$ The Jacobian is:

$$Df = \begin{bmatrix} 1-3x_2 & -3x_1 \\ 1 & 1 \end{bmatrix} \quad Df(2,1) = \begin{bmatrix} -2 & -6 \\ 1 & 1 \end{bmatrix}$$

   Therefore, the linearization is:

$$\boldsymbol{y} - \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 - 1 \end{bmatrix}$$

2. Linearize the differential equation at the point $(-1,2)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \sin(\pi x_2) \\ x_1 - x_1^3 \end{bmatrix}$$

   SOLUTION: We let $f(x_1,x_2) = \begin{bmatrix} \sin(\pi x_2) \\ x_1 - x_1^3 \end{bmatrix}$, and $f(-1,2) = [0,0]^T$. Compute the Jacobian:

$$Df = \begin{bmatrix} 0 & \pi\cos(\pi x_2) \\ 1-3x_1^2 & 0 \end{bmatrix} \quad Df(-1,2) = \begin{bmatrix} 0 & \pi \\ -2 & 0 \end{bmatrix}$$

   Now the linearization is given by:

$$\begin{bmatrix} x_1' - 0 \\ x_2' - 0 \end{bmatrix} = \begin{bmatrix} 0 & \pi \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 + 1 \\ x_2 - 2 \end{bmatrix}$$

3. Linearize the differential equation at the point $(2,1/3)$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1(1-3x_2) \\ x_2(x_1-2) \end{bmatrix}$$

   SOLUTION: We let $f(x_1,x_2) = \begin{bmatrix} 5x_1(1-3x_2) \\ x_2(x_1-2) \end{bmatrix}$, and $f(2,1/3) = [0,0]^T$. Compute the Jacobian:

$$Df = \begin{bmatrix} 5-15x_2 & -15x_1 \\ x_2 & x_1 - 2 \end{bmatrix} \quad Df(2,1/3) = \begin{bmatrix} 0 & -30 \\ 1/3 & 0 \end{bmatrix}$$

   Now the linearization is given by:

$$\begin{bmatrix} x_1' - 0 \\ x_2' - 0 \end{bmatrix} = \begin{bmatrix} 0 & -30 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} x_1 + 1 \\ x_2 - 2 \end{bmatrix}$$

4. Use the Poincaré Classification Diagram to state the behavior of the previous system at the point $(2,1/3)$:
   We use the matrix coming from the Jacobian:

$$\begin{bmatrix} 0 & -30 \\ 1/3 & 0 \end{bmatrix}$$

   In this case, $\mathrm{Tr}(A) = 0$, $\det(A) = 10$, $\Delta = -40$. Thus, the behavior of the nonlinear system close to the point $(2,1/3)$ is similar to a CENTER.

The Equilibrium Solutions for Systems

Earlier, for $y' = f(y)$, we said that the equilibria were found where $y' = 0$, or equivalently, where $f(y) = 0$. The same is true for systems: The equilibria for the system $\boldsymbol{x}' = F(\boldsymbol{x})$ are where $F(\boldsymbol{x}) = \boldsymbol{0}$. Note that this says we have to solve two equations simultaneously.

EXAMPLE: Find the equilibria for the following system:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} (2+x_1)(x_2-x_1) \\ (4-x_1)(x_1+x_2) \end{bmatrix}$$

To solve the first equation, $(2+x_1)(x_2-x_1) = 0$, we must have that either $x_1 = -2$ or $x_1 = x_2$. Take each of these, and look at the second equation:

If $x_1 = -2$, the second equation becomes: $6(-2+x_2) = 0$, so $x_2 = 2$

If $x_1 = x_2$, the second equation becomes: $(4-x_1)2x_1 = 0$, so $x_1 = x_2 = 0$ or $x_1 = x_2 = 4$.

Altogether, there are three equilibria: $(-2,2)$, $(0,0)$, $(4,4)$.

EXAMPLE: Find the equilibria for the following system:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 - x_1^2 - x_1x_2 \\ 3x_2 - x_1x_2 - 2x_2^2 \end{bmatrix}$$

To solve the first equation, $x_1(1-x_1-x_2) = 0$, we must have that either $x_1 = 0$ or $x_1 = 1-x_2$. Take each of these, and look at the second equation:

If $x_1 = 0$, the second equation becomes: $x_2(3-2x_2) = 0$, so $x_2 = 0$ or $x_2 = 3/2$

If $x_1 = 1-x_2$, the second equation becomes (this has been simplified): $x_2(2-x_2) = 0$, so $x_2 = 0$ (and $x_1 = 1$) or $x_2 = 2$ (and $x_1 = -1$).

Altogether, there are four equilibria: $(0,0)$, $(0,3/2)$, $(1,0)$ and $(-1,2)$

THE ANALYSIS OF NONLINEAR SYSTEMS OF D.E.s

In general, we will not be able to solve a nonlinear system of differential equations. The method for analyzing a nonlinear system is to analyze the behavior close to the equilibria (much like we did with $y' = f(y)$). Thus, the analysis of a nonlinear system will proceed as follows:

- Find the equilibria.
- About each equilibria, convert the nonlinear system, $\boldsymbol{x}' = F(\boldsymbol{x})$ into a (locally) linear system $\boldsymbol{x}' = A\boldsymbol{x}$. Classify the behavior here using the Poincaré classification diagram.

EXAMPLE: Analyze the following system by linearization about the equilibria:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 - x_1^2 - x_1x_2 \\ 3x_2 - x_1x_2 - 2x_2^2 \end{bmatrix}$$

Altogether, there were four equilibria: $(0,0)$, $(0,3/2)$, $(1,0)$ and $(-1,2)$ found previously. We will linearize the differential equation at each of them, so we'll need the Jacobian matrix:

$$Df(x_1,x_2) = \begin{bmatrix} 1-2x_1-x_2 & -x_1 \\ -x_2 & 3-x_1-4x_2 \end{bmatrix}$$

Now we go through each equilibria and use Poincaré:

- At $(0,0)$, $Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. The trace is 4, the determinant is 3, and the discriminant is $4^2 - 12 = 4$. The origin is a SOURCE.
- At $(0,3/2)$, $Df(0,3/2) = \begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$. The trace is $-7/2$, the determinant is $3/2$, and the discriminant is $25/2$. The equilibrium at $(0,3/2)$ is a SINK.

- At $(1,0)$, $Df(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$. The trace is 1, the determinant is -2, and the discriminant is 9. There is a SADDLE at $(1,0)$.
- At $(-1,2)$, $Df(-1,2) = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$. The trace is -3, the determinant is -2, and the discriminant is 17. There is a SADDLE at $(-1,2)$.

Attached is a Maple plot of the direction field- See if you can see the source, sink and saddles at the equilibria we found.

Exercises

1. For each linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where A is as given below, use the Poincaré classification diagram to analyze how the behavior at the origin changes with α:

$$\begin{bmatrix} 4 & \alpha \\ 8 & -6 \end{bmatrix}, \quad \begin{bmatrix} \alpha & 10 \\ -1 & -4 \end{bmatrix}, \quad \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$$

2. For each nonlinear system, (i) Find the equilibria, (ii) at each equilibrium, linearize the system and classify using the Poincaré Diagram.

   (a)
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1-y \\ x^2 - y^2 \end{bmatrix}$$

   (b) (The van der Pol Equation)
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ (1-x^2)y - x \end{bmatrix}$$

   Additionally, you should plot the direction field and some sample solutions in Maple (See the Competing Species Maple Worksheet that is online).

   (c)
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -(x-y)(1-x-y) \\ x(2+y) \end{bmatrix}$$

   (d)
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - \frac{1}{2}xy \\ -\frac{3}{4}y + \frac{1}{4}xy \end{bmatrix}$$

   Additional question for this system: This is a Predator-Prey equation. Which of x, y is the predator, and which is the prey? Why?

3. The following is a model for the populations of two competing species (competing for the same resources).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x(1-\frac{1}{2}x) - xy \\ 3y(1-\frac{1}{3}y) - 2xy \end{bmatrix}$$

   (a) Does this model seem reasonable? Think about what happens to one species in the absence of the other. Also, consider how the interaction terms (xy terms) effects the change in populations.
   (b) Find the equilibria.
   (c) Classify the equilibria by linearizing the differential equation at each one.