Supplementary Exercises, Section 4.1

1. For each differential equation below, write down its associated operator.

   (a) $y' = 3xy + x^2$ (b) $y' = -3y + 5y^2$ (c) $xyy' + y^2 - 2x = 0$

2. Below, an operator is defined and a function is given. Apply the operator to the function:

   (a) $F(f(t)) = \int_0^x f(t)\mathrm{e}^{-t}\, dt$, $f(t) = t^2$
   (b) $F(y(t)) = ty' + 3y$, $y(t) = 3t^{-3}$
   (c) $F(y(t)) = y'' - 3y' + 2y$, $y(t) = sin(t)$

3. For each of the previous operators, determine whether it was a linear operator using the definition of a linear operator.

   Definition: The kernel of the operator L is the set of functions so that $L(y) = 0$.

4. Below you are given an operator and a function. Determine if the function is in the kernel of the operator.

   (a) $F(y(t)) = y'' - 3y' + 2y$, $y(t) = C\mathrm{e}^{2t}$ (C is an arbitrary constant)
   (b) $F(y(t)) = x^2y'' - 2xy' - 4y$, $y = Cx^4$
   (c) $F(y(t)) = x^3y''' - 2xy' + 4y$, $y = Cx^2$

   Definition: A set of functions $f_1, f_2, \ldots, f_n$ are said to be *linear independent* if the only solution to

   $$c_1f_1 + c_2f_2 + \ldots + c_nf_n = 0$$

   is $c_1 = c_2 = \ldots = c_n = 0$. NOTE: The solution we find must be valid *for all values of the domain.*

   Example: Show that t and t^2 are linearly independent functions on any interval.

   Solution: We are solving $c_1t + c_2t^2 = 0$ for all t in some interval. Since this equation must be true for all t, in particular, it must be true for specific values of t. For $t = 1$, we must have $c_1 + c_2 = 0$. For $t = -1$, we must have $-c_1 + c_2 = 0$. Putting these together, $c_1 = c_2 = 0$ is the only solution.

5. For each set of functions, determine if the set is linearly independent. If it depends on the interval, state that as well.

   (a) $1, t, t^2$
   (b) t and $|t|$

(c) e^t and $2e^t$

(d) $\sin(2t)$, $\sin(3t)$

Definition: The determinant of an array. For a 2×2 array, the determinant is defined to be:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 array, you might recall this from *cross products*:

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = i(bf - ec) - j(af - cd) + k(ae - bd)$$

6. Compute each of the given determinants:

$$\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} -2 & 1 & 0 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

7. Compute the Wronskian of each set of functions:

(a) $y_1 = e^x$, $y_2 = xe^x$

(b) $y_1 = t$, $y_2 = |t|$

(c) $y_1 = \sin(t)$, $y_2 = \cos(t)$

(d) $y_1 = e^{2x}$, $y_2 = e^{-x}$, $y_3 = e^x$

8. The hyperbolic sine and cosine are defined as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

These are used when solutions to differential equations involve the exponential. There are some nice properties of the hyperbolic functions- verify these:

- $\frac{d}{dx}(\sinh(x)) = \cosh(x)$, $\frac{d}{dx}(\cosh(x)) = \sinh(x)$
- $\cosh^2(x) - \sinh^2(x) = 1$