**Summary Sheet: Complex Numbers**

**Definition:** A complex number is anything of the form $a + bi$, where $a, b$ are real and $i = \sqrt{-1}$.

- The real numbers are a subset of the complex numbers (set $b = 0$ to get the real numbers). Since $i = \sqrt{-1}$, we can easily compute the powers of $i$: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc.
- The conjugate of $a + bi$ is $a - bi$. Notation: $\overline{a + bi} = a - bi$
- The "real part" of $a+bi$ is $a$, the imaginary part of $a+bi$ is $b$. Notation: $\mathrm{Re}(a+bi) = a$, $\mathrm{Im}(a + bi) = b$.
- The modulus, or size, of a complex number: $|a + bi| = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$.
- Complex numbers can be plotted in two dimensions. The complex number $a + bi$ is plotted as $(a, b)$. Notice that $|a + bi|$ is the distance from $(a, b)$ to $(0, 0)$.
- Euler's Formula: $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$. From this, we get one of the most elegant equations of mathematics:

  $$\mathrm{e}^{i\pi} + 1 = 0$$

  Note that this gives all of the big constants of mathematics in a single beautiful relationship.

**Complex Arithmetic** is very similar to polynomial arithmetic:

1. Addition/subtraction: $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$
2. Multiplication: $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$
3. Division: Multiply by the conjugate-

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

**The Polar Form of a Complex Number**

We already know the polar form of an ordered pair, $(a, b)$. In calculus, we learned that $a = R\cos(\theta)$, $b = R\sin(\theta)$, where $R = \sqrt{a^2 + b^2}$ and $\theta$ is the (four-quadrant) angle that a line from $(0, 0)$ to $(a, b)$ would make with the positive $x-$axis. Therefore,

$$a + bi \Rightarrow (a, b) \Rightarrow (R\cos(\theta), R\sin(\theta)) \Rightarrow R(\cos(\theta) + i\sin(\theta))$$

We can therefore write $a + bi$ as $R(\cos(\theta) + i\sin(\theta)$. Using Euler's formula, we can write $a + bi$ simply as $R\mathrm{e}^{i\theta}$. Using this, we can define powers and logs using $i$:

- $\ln(a + bi) = \ln(R\mathrm{e}^{i\theta}) = \ln(R) + i\theta$
- $a^{\beta i} = \mathrm{e}^{i\ln(a)}$

We have extended logarithms to negative real numbers- Note that $\ln(-1) = \ln(1) + i \cdot \pi$, so $\ln(-1) = \pi i$. You can also do things like: $i^i = \mathrm{e}^{-\pi/2}$

**Rules of Exponents Still Apply** For example, $\mathrm{e}^{a+bi} = \mathrm{e}^a\mathrm{e}^{bi}$, etc.