Practice Sheet: Complex Functions in Differential Equations

1. Solve for λ:

   (a) $\lambda^2 - 6\lambda + 13 = 0$

   $$\lambda = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

   (b) $2\lambda^3 - 8\lambda^2 + 12\lambda - 8 = 0$ (Hint: One solution is $\lambda = 2$). Use long division to get:

   $$2\lambda^3 - 8\lambda^2 + 12\lambda - 8 = 2(\lambda - 2)(\lambda^2 - 2\lambda + 2) = 0$$

   Use the quadratic formula on the last expression: $\lambda = 2, 1 \pm i$.

   (c) $\lambda^3 + \lambda^2 + 4\lambda + 4 = 0$ (Hint: Try to factor in groups)

   $$\lambda^2(\lambda + 1) + 4(\lambda + 1) = (\lambda + 1)(\lambda^2 + 4) = 0$$

   so that $\lambda = -1, \pm 2i$

   (d) $\lambda^4 - 18\lambda^2 + 81 = 0$ In this case, we can use the quadratic formula (or factoring directly) to solve for λ^2:

   $$(\lambda^2 - 9)^2 = 0 \Rightarrow \lambda = 3, 3, -3, -3$$

2. Simplify each of the following to the form $a + bi$:

   (a) $(3 + 5i)(2 - 3i) = 21 + i$

   (b) $(3 - 2i)\overline{3 - 2i} = 13$

   (c) $|-2 + i| = \sqrt{5}$

   (d) $(2 - i)/(1 + 3i) = -\frac{1}{10} - \frac{7}{10}\, i$

   (e) $(1 + i)^{2+3i} = (1 + i)^2(1 + i)^{3i} = 2i(1 + i)^{3i}$ and

   $$(1 + i)^{3i} = \mathrm{e}^{3i \ln(1+i)} = \mathrm{e}^{3i(\sqrt{2} + i\pi/4)} = \mathrm{e}^{-3\pi/4 + 3\sqrt{2}i} = \mathrm{e}^{-3\pi/4}\left(\cos(3\sqrt{2}) + i \sin(3\sqrt{2})\right)$$

   Put the pieces together, and:

   $$-2\mathrm{e}^{-3\pi/4} \sin(3\sqrt{2}) + i \cdot 2\mathrm{e}^{-3\pi/4} \cos(3\sqrt{2})$$

   (f) $\ln\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

   Note that the angle corresponding to $(-1/2, \sqrt{3}/2)$ is given by $\pi - \pi/3 = 2\pi/3$ (the point is in the second quadrant). Therefore,

   $$\ln\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \ln(1) + \frac{2\pi}{3}\, i = \frac{2\pi}{3}\, i$$

   (g) $\mathrm{e}^{2+i}\mathrm{e}^{-3i} = \mathrm{e}^{2-2i} = \mathrm{e}^2\left(\cos(2) - i\sin(2)\right)$

   (h) $\ln(-5) = \ln(5) + i\pi$

   (i) $\mathrm{e}^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = \cos(\pi) - i\sin(\pi) = 1$

3. Write each complex number in its polar form:

   (a) $1 - \sqrt{3}i$ $R = \sqrt{1 + 3} = 2$, and θ is in the 4th quadrant. From the $1 - 2 - \sqrt{3}$ triangle, we see that $\theta = -\frac{\pi}{3}$.

   $$1 - \sqrt{3}i = 2\mathrm{e}^{-i\pi/3} \text{ or } 2\mathrm{e}^{i\,5\pi/3}$$

   (b) 3 The positive real numbers are already in polar form! $R = 3, \theta = 0$.

(c) $-5-3i$ (Use a calculator for θ)
Using the tangent, $\tan^{-1}(3/5) \approx 0.5404$. Note that this angle is in the first quadrant rather than the fourth. Therefore, $\theta = \pi + 0.5404 \approx 3.682$ $R = \sqrt{25+9} = \sqrt{34}$, so

$$-5-3i \approx \sqrt{34}\mathrm{e}^{3.682\,i}$$

(d) $-i = \mathrm{e}^{3\pi/2\,i}$ or $\mathrm{e}^{-\pi/2\,i}$

4. Show, using Euler's Formula and treating i as a constant, that:

$$\frac{d}{dt}\left(\mathrm{e}^{(\alpha+\beta i)t}\right) = (\alpha+\beta i)\mathrm{e}^{(\alpha+\beta i)t}$$

Doing this the long way and using the product rule:

$$\frac{d}{dt}\left(\mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)\right) =$$

$$\alpha\mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) + \mathrm{e}^{\alpha t}\left(-\beta\sin(\beta t) + i\beta\cos(\beta t)\right) =$$

$$\alpha\mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) + \beta i\mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) = (\alpha+\beta i)\mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$$

which gives us our result.

5. Verify our in-class remarks that, if $y = C_1\mathrm{e}^{(\alpha+\beta i)t} + C_2\mathrm{e}^{(\alpha-\beta i)t}$, and $y(0) = 1$, $y'(0) = \alpha$, then $y = \mathrm{e}^{\alpha t}\cos(\beta t)$. We get the two equations:

$$y(0) = 1 \qquad \Rightarrow \qquad 1 = C_1 + C_2 \qquad 1 - C_1 = C_2$$

and

$$y'(0) = \alpha \qquad \Rightarrow \qquad \alpha = (\alpha+\beta i)C_1 + (\alpha-\beta i)C_2$$

Putting these together,

$$\alpha = (\alpha+\beta i)C_1 + (\alpha-\beta i)(1-C_1) \qquad \Rightarrow \qquad \alpha = (\alpha+\beta i)C_1 + \alpha - \beta i - (\alpha-\beta i)C_1 \qquad \Rightarrow$$

$$\beta i = (\alpha+\beta i - \alpha + \beta i)C_1 \qquad C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$$

Now,

$$\frac{1}{2}\mathrm{e}^{(\alpha+\beta i)t} + \frac{1}{2}\mathrm{e}^{(\alpha-\beta i)t} = \frac{1}{2}\mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t) + \cos(\beta t) - i\sin(\beta t)\right) = \mathrm{e}^{\alpha t}\cos(\beta t)$$

6. Similarly, verify that, if $y = C_1\mathrm{e}^{(\alpha+\beta i)t} + C_2\mathrm{e}^{(\alpha-\beta i)t}$, and $y(0) = 0$, $y'(0) = \beta$, then $y = \mathrm{e}^{\alpha t}\sin(\beta t)$. We get the two equations:

$$y(0) = 0 \qquad \Rightarrow \qquad 0 = C_1 + C_2 \qquad -C_1 = C_2$$

and

$$y'(0) = \beta \qquad \Rightarrow \qquad \beta = (\alpha+\beta i)C_1 + (\alpha-\beta i)C_2$$

Putting these together,

$$\beta = (\alpha+\beta i)C_1 + (\alpha-\beta i)(-C_1) \qquad \Rightarrow \qquad \beta = (\alpha+\beta i - \alpha + \beta i)C_1 \qquad \Rightarrow C_1 = \frac{1}{2i} = -\frac{i}{2}$$

Now,

$$\frac{-i}{2}\mathrm{e}^{(\alpha+\beta i)t} + \frac{i}{2}\mathrm{e}^{(\alpha-\beta i)t} = \frac{i}{2}\mathrm{e}^{\alpha t}\left(-\mathrm{e}^{\beta i t} + \mathrm{e}^{-\beta i t}\right) =$$

$$\frac{i}{2}\mathrm{e}^{\alpha t}\left(-\cos(\beta t) - i\sin(\beta t) + i\cos(\beta t) - i\sin(\beta t)\right) = \mathrm{e}^{\alpha t}\sin(\beta t)$$