An Example in Detail

Are t and $|t|$ linearly independent? By the definition, functions y_1 and y_2 are linearly independent if the **only** solution to:

$$c_1 y_1 + c_2 y_2 = 0$$

is $c_1 = c_2 = 0$ for all t.
In this case, if we restrict $t > 0$, then $|t| = t$, and so $c_1 = -c_2$ gives an infinite number of solutions that are nonzero. In particular, $c_1 = 1$ and $c_2 = -1$. That is, for $t > 0$,

$$c_1 t + c_2 |t| = t - |t| = t - t = 0$$

Conclusion: If $t < 0$, the functions are linearly dependent.
On the other hand, if $t < 0$, then $|t| = -t$ and $c_1 = c_2$ will give an infinite number of nonzero solutions. In particular, we could take $c_1 = c_2 = 1$. That is, for $t < 0$,

$$c_1 t + c_2 |t| = t + |t| = t + (-t) = 0$$

Conclusion: If $t > 0$, the functions are linearly dependent.
What happens if our interval contains BOTH positive and negative numbers, like $[-a, b]$? We can't use the first set of constants, $c_1 = 1$ and $c_2 = -1$, because:

$$t - |t| = \begin{cases} 0 & \text{if } 0 \le t \le b \\ 2t & \text{if } -a \le t < 0 \end{cases}$$

Similarly, we can't use the second set of constant, $c_1 = c_2 = 1$, because:

$$t + |t| = \begin{cases} 2t & \text{if } 0 \le t \le b \\ 0 & \text{if } -a \le t < 0 \end{cases}$$

If our interval contains both positive and negative numbers, it must include two numbers like $-a, a$ $(a > 0)$, then:

$$\text{If } t = a \Rightarrow ac_1 + ac_2 = 0 \qquad \text{If } t = -a \Rightarrow -ac_1 + ac_2 = 0$$

Putting these together, we see that $c_1 = c_2 = 0$.
Conclusion: If the interval contains both positive and negative numbers, the functions t and $|t|$ are linearly independent.

By the way, these functions also can create a little confusion with the Wronskian. The Wronskian of these functions:
If $t > 0$:

$$\begin{vmatrix} t & t \\ 1 & 1 \end{vmatrix} = t - t = 0$$

If $t < 0$,

$$\begin{vmatrix} t & -t \\ 1 & -1 \end{vmatrix} = -t + t = 0$$

We might conclude therefore, that the Wronskian is zero for every $t \neq 0$, and therefore the functions are linearly independent. Why is this not true?