Math 244, Spring 2005: Exam 1 Review

1. Create a model of a population bounded by an environmental equilibrium. Your assumptions are that the differential equation is autonomous, that $y = 0$ is an unstable equilibrium and $y = A$ is a stable equilibrium. (Hint: Draw a possible graph of $y' = f(y)$ as a phase plot, then get a possible equation for your graph). Once you've obtained your differential equation, solve it. Does your formula for the solution behave as it should as $t \to \infty$? Explain.

2. Suppose you have a tank of brine containing 300 gallons of water with a concentration of 1/6 pounds of salt per gallon. There is brine pouring into the tank at a rate of 3 gallons per minute, and it contains 2 pounds of salt per gallon. The well-mixed solution leaves at the same rate, 3 gallons per minute. Write the initial value problem for the amount of salt in the tank at time t, and solve it. What is the concentration of salt in the tank as $t \to \infty$? (and does that make sense?).

3. Same as the previous problem, but suppose that the brine is leaving the tank at 2 gallons per minute. Write and solve the new initial value problem. If the tank could hold an infinite amount of brine, is there a limit to the concentration of salt in the tank?

4. Attached is a set of four direction fields plotted from Maple. Classify each as coming from a differential equation that is of the form: (a) $y' = f(y)$, (b) $y' = f(x)$, (c) A D.E. with a transient solution. (d) None of these.

5. Classify each D.E. by its type- (i) Linear, (ii) Separable, (iii) Exact, (iv) Homogeneous, (v) Bernoulli. (NOTE: There may be more than one answer for each). Do not solve.

   (a) $\dfrac{dy}{dx} = \dfrac{x - y}{x}$

   (b) $\dfrac{y}{x^2}\dfrac{dy}{dx} + e^{2x+y^2} = 0$

   (c) $(x + 1)\dfrac{dy}{dx} = -y + 10$

   (d) $\dfrac{dy}{dx} = \dfrac{1}{x(y - x)}$

   (e) $\dfrac{dy}{dx} = \dfrac{y^2 + y}{x^2 + x}$

   (f) $\dfrac{dy}{dx} - 5y = y^2$

   (g) $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}$

   (h) $y\,dx = (y - xy^2)\,dy$

   (i) $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x} + 1$

(j) $\left(x^2 + \frac{2y}{x}\right) dx = (3 - \ln(x^2))\, dy$

6. True or False, and give a short explanation. If a statement is sometimes true and sometimes false, then it is false- but give a reason.

   (a) In the direction field for $y' = f(x, y)$, two solution curves cannot intersect.

   (b) All separable equations are also exact.

   (c) A solution to $y' = \sin(y)$ is probably periodic.

   (d) The particular solution to $y' + y = 3\sin(t) - 1$ will probably be of the form: $y = A\sin(t) + B\cos(t) + C$

   (e) The homogeneous part of the solution to $y' + y = 3\sin(t) + C$ is Ae^{-t}.

   (f) If our differential equation is: $y' + \frac{2}{\sqrt{x-3}}y = 4$, then we can predict that the solution will be valid on the entire interval $x > 3$.

   (g) If our differential equation is: $y' + \frac{2}{\sqrt{x-3}}y^2 = 4$, then we can predict that the interval on which our solution is valid will be $x > 3$.

7. Find values of k for which the IVP: $xy' - 4y = 0$, $y(0) = k$ has (i) No solution, (ii) An infinite number of solutions. Does this violate the Existence and Uniqueness Theorem (explain)?

8. Find values of k so that $y = e^{kt}$ is a solution to $y'' + y' - 2y = 0$.

9. Find values of n so that $y = x^n$ is a solution to: $x^2y'' - 7xy' + 15y = 0$.

10. Find A, B and C so that $y = A\sin(t) + B\cos(t)$ is a particular solution to $y' + y = 3\sin(t) - 1$ (Also see Problem 6(c))

11. Given Problem 6(c, d) and your previous answer, what is the full general solution to $y' + y = 3\sin(t) - 1$? Verify this by solving it directly.

12. Solve, and give the interval I on which the solution is valid: $\frac{dy}{dx} = \sqrt{1 - y^2}$

13. Solve:

    (a) $\frac{dP}{dt} + 2tP = P + 4t - 2$

    (b) **CHANGE TO:** $(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$

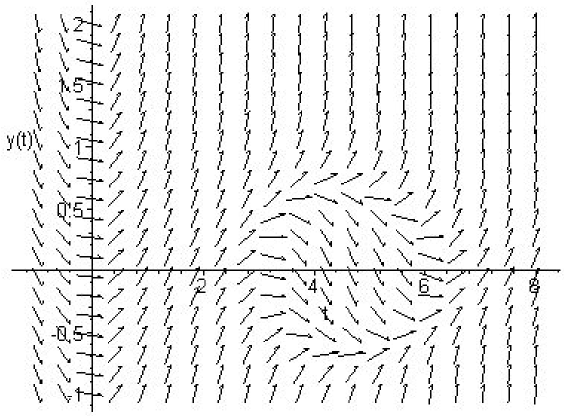
    (c) $(y^2 + 1)\, dx - (y\sec^2(x))\, dy = 0$

    (d) $(6x+1)y^2\frac{dy}{dx} + 3x^2 + 2y^3 = 0$

    (e) $xy^2y' = y^3 - x^3$

    (f) $y' + y = f(x)$, $y(0) = 1$, where $f(x) = \begin{cases} -1 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

14. If $y' + \frac{2}{x^2 - 1}y = 3x + 2$, $y(0) = y_0$, what is the largest interval on which the solution is defined?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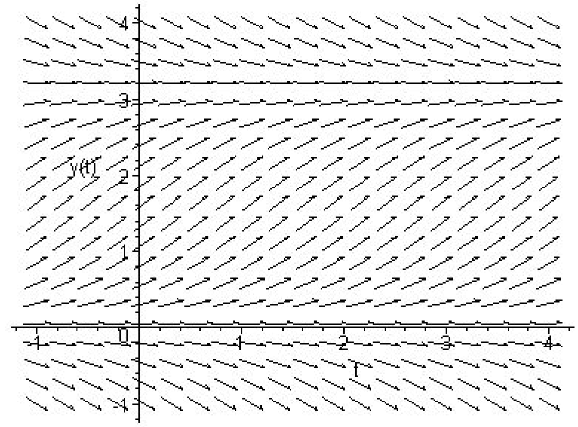
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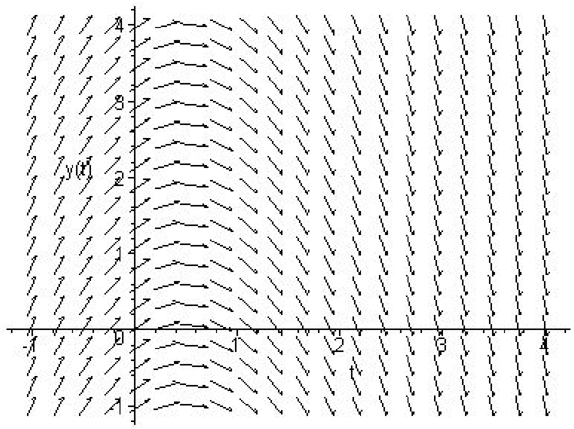
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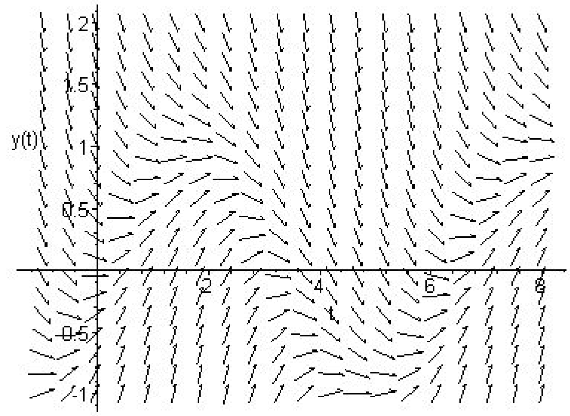
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