Math 244, Spring 2005: Exam 1 Review

1. Create a model of a population bounded by an environmental equilibrium. Your assumptions are that the differential equation is autonomous, that $y = 0$ is an unstable equilibrium and $y = A$ is a stable equilibrium. (Hint: Draw a possible graph of $y' = f(y)$ as a phase plot, then get a possible equation for your graph). Once you've obtained your differential equation, solve it. Does your formula for the solution behave as it should as $t \to \infty$? Explain.

   There are many possible solutions. The most straightforward would be to take y' to be an upside down parabola with intercept $y = 0$ and $y = A$. That would give: $y' = ky(A - y)$, where you could choose k to be some positive constant. We'll assume here that $k = 1$.

   This is a separable equation, with equilibrium solutions at $y = 0$ and $y = A$. Solving for the general solution:

$$\int \frac{1}{y(A-y)}\,dy = \int 1\,dx \Rightarrow \frac{1}{A}\int \frac{1}{y}\,dy + \frac{1}{A}\int \frac{1}{A-y}\,dy = x + C_1$$

$$\frac{1}{A}\ln|y| - \frac{1}{A}\ln|A-y| = x + C_1 \Rightarrow \ln\left|\frac{y}{A-y}\right| = Ax + C_2 \Rightarrow \frac{y}{A-y} = C_3 e^{Ax}$$

$$y = AC_3e^{Ax} - C_3e^{Ax}y \Rightarrow y(1 + C_3e^{Ax}) = AC_3e^{Ax} \Rightarrow y = \frac{AC_3e^{Ax}}{1 + C_3e^{Ax}}$$

   As $x \to \infty$, $y \to A$. In the case that y_0 is negative, we have a vertical asymptote. For example, if $y_0 = -1$, then:

$$-1 = \frac{AC_3}{1 + C_3} \Rightarrow C_3 = \frac{-1}{1 + A}$$

   and there will be a vertical asymptote where

$$1 - \frac{1}{1+A}e^{Ax} = 0 \Rightarrow x = \frac{\ln(1 + A)}{A}$$

2. Suppose you have a tank of brine containing 300 gallons of water with a concentration of 1/6 pounds of salt per gallon. There is brine pouring into the tank at a rate of 3 gallons per minute, and it contains 2 pounds of salt per gallon. The well-mixed solution leaves at the same rate, 3 gallons per minute. Write the initial value problem for the amount of salt in the tank at time t, and solve it. What is the concentration of salt in the tank as $t \to \infty$? (and does that make sense?).

$$\frac{dA}{dt} = 3 \cdot 2 - 3 \cdot \frac{A}{300} \Rightarrow A' + \frac{1}{100}A = 6, \quad A(0) = 50$$

   This is a linear DE, so solve using an integrating factor of $e^{(1/100)t}$:

$$A = 600 + Ce^{-(1/100)t} \Rightarrow A = 600 - 550e^{-(1/100)t}$$

   As $t \to \infty$, $A \to 600$ pounds, which corresponds to a concentration of 2 pounds per gallon, which is the incoming concentration of salt.

3. Same as the previous problem, but suppose that the brine is leaving the tank at 2 gallons per minute. Write and solve the new initial value problem. If the tank could hold an infinite amount of brine, is there a limit to the concentration of salt in the tank?

$$\frac{dA}{dt} = 3 \cdot 2 - 2 \cdot \frac{A}{300 + t} \Rightarrow A' + \frac{2}{300 + t}A = 6, \quad A(0) = 50$$

   Use an integrating factor of $e^{2\ln(300+t)} = (300 + t)^2$,

$$((300 + t)^2A)' = 6(300 + t^2) \Rightarrow (300 + t)^2A = 2(300 + t)^3 + C$$

$$A = 2(300 + t) + \frac{C}{(300 + t)^2}\ A(0) = 50 \Rightarrow A = 2(300 + t) - \frac{550 \cdot 300^2}{(300 + t)^2}$$

Note that this has a transient term, which gets small very quickly. This means that the amount of salt at time t is approximately $2(300+t)$, which is the incoming concentration times the number of gallons of water. Formally, the concentration of salt in the tank at time t:

$$\frac{A}{300+t} = 2 - \frac{C}{(300+t)^3}$$

so the limiting concentration is 2 pounds of salt per gallon.

4. Below is a set of four direction fields plotted from Maple. Classify each as coming from a differential equation that is of the form: (a) $y' = f(y)$, (b) $y' = f(x)$, (c) A D.E. with a transient solution. (d) Neither of these.

   Counting the pictures as: $\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$, (1)- None of the above, (2)- Autonomous, (3)- $y' = f(x)$, (4)=Transient

5. Classify each D.E. by its type- (i) Linear, (ii) Separable, (iii) Exact, (iv) Homogeneous, (v) Bernoulli. (NOTE: There may be more than one answer for each). Do not solve.

   In general, all separable equations are also exact.

   (a) $\frac{dy}{dx} = \frac{x-y}{x}$ Homogeneous, Linear

   $$y' = 1 - \frac{y}{x} \Rightarrow y' + \frac{1}{x}y = 1$$

   (b) $\frac{y}{x^2}\frac{dy}{dx} + e^{2x+y^2} = 0$ Separable, Exact

   (c) $(x+1)\frac{dy}{dx} = -y + 10$ Separable, Exact, Linear

   (d) $\frac{dy}{dx} = \frac{1}{x(y-x)}$ None of these with y as a function of x, but Bernoulli in x.

   (e) $\frac{dy}{dx} = \frac{y^2+y}{x^2+x}$ Separable, Exact

   (f) $\frac{dy}{dx} - 5y = y^2$ Bernoulli, Separable, Exact

   (g) $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$ Bernoulli, Homogeneous

   (h) $y\,dx = (y - xy^2)\,dy$ None of these with y as a function of x, but linear in x.

   (i) $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} + 1$ Homogeneous

   (j) $\left(x^2 + \frac{2y}{x}\right) dx = (3 - \ln(x^2))\,dy$ Exact

6. True or False, and give a short explanation. If a statement is sometimes true and sometimes false, then it is false- but give a reason.

   (a) In the direction field for $y' = f(x,y)$, two solution curves cannot intersect. True, if f satisfies the Existence and Uniqueness theorem at all (x,y): That is, f is continuous and $\frac{\partial f}{\partial y}$ is continuous.

   (b) All separable equations are also exact. True- If separable, then you can write the equation as:

   $$G(x)\,dx + F(y)\,dy = 0$$

   and $\frac{\partial G}{\partial y} = \frac{\partial F}{\partial x} = 0$.

(c) A solution to $y' = \sin(y)$ is probably periodic. False. No solution to an autonomous equation can be periodic. That is because in the direction field, all the slopes must be the same along any horizontal line (the derivative does not depend on x).

(d) The particular solution to $y' + y = 3\sin(t) - 1$ will probably be of the form: $y = A\sin(t) + B\cos(t) + C$

True. Since the derivatives of sines and cosines are sines and cosines, and in this equation we are simply adding the derivatives with the original function. We can also see that $C = -1$.

(e) The homogeneous part of the solution to $y' + y = 3\sin(t) + C$ is $A\mathrm{e}^{-t}$.

True. The homogeneous part of the solution is the solution to $y' + y = 0$.

(f) If our differential equation is: $y' + \frac{2}{\sqrt{x-3}}y = 4$, then we can predict that the solution will be valid on the entire interval $x > 3$.

True. This is a linear differential equation- The E&U Theorem is much stronger for this type. Recall the setup: If

$$y' + p(x)y = f(x)$$

and f, p were continuous on an interval I containing x_0, then the solution is unique, and the interval on which the solution is valid is I.

(g) If our differential equation is: $y' + \frac{2}{\sqrt{x-3}}y^2 = 4$, then we can predict that the interval on which our solution is valid will be $x > 3$.

False. We would need to use the general E&U Theorem in this case. In that theorem, we could not predict how large the interval would be.

7. Find values of k for which the IVP: $xy' - 4y = 0$, $y(0) = k$ has (i) No solution, (ii) An infinite number of solutions. Does this violate the Existence and Uniqueness Theorem (explain)?

$$y' - \frac{4}{x}y = 0 \quad \Rightarrow \quad y = Cx^4$$

so if $k = 0$, C can be any number (so there is an infinite number of solutions). If $k \neq 0$, if we try to put in the initial condition, we would have: $k = C \cdot 0^4$, or $k = 0$. Therefore, if $k \neq 0$, there is no solution.

This does not violate the E&U Theorem since $-\frac{4}{x}$ is not continuous at $x = 0$.

8. Find values of k so that $y = \mathrm{e}^{kt}$ is a solution to $y'' + y' - 2y = 0$.

Substitute: $y = \mathrm{e}^{kt}, y' = k\mathrm{e}^{kt}, y'' = k^2\mathrm{e}^{kt}$ to get:

$$\mathrm{e}^{kt}\left(k^2 + k - 2\right) = 0$$

There is no solution to $\mathrm{e}^{kt} = 0$, so the only solutions are from $k^2 + k - 2 = 0$, or $(k-1)(k+2) = 0$. So $k = 1, -2$.

9. Find values of n so that $y = x^n$ is a solution to: $x^2y'' - 7xy' + 15y = 0$.

Substitute: $y = x^n$, $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}$ to get:

$$n(n-1)x^n - 7nx^n + 15x^n = 0 \quad \Rightarrow \quad x^n(n^2 - 8n + 15) = 0$$

Either $x^n = 0$ or $(n-3)(n-5) = 0$. Note that when we say $x^n = 0$, the n is the unknown, and x can vary- No solution from this, although $y = 0$ is a solution to the original. That leaves $n = 3, 5$.

10. Find A, B and C so that $y = A\sin(t) + B\cos(t)$ is a particular solution to $y' + y = 3\sin(t) - 1$ (Also see Problem 6(c))

Substitute: $y = A\sin(t) + B\cos(t) + C$, $y' = A\cos(t) - B\sin(t)$ into the D.E. to get:

$$(A - B)\sin(t) + (A + B)\cos(t) + C = 3\sin(t) - 1$$

Therefore, $A - B = 3$, $A + B = 0$, and $C = -1$. From the first two equations, we get $A = 3/2$, $B = -3/2$.

11. Given Problem 6(c, d) and your previous answer, what is the full general solution to $y'+y = 3\sin(t)-1$? Verify this by solving it directly.

The general solution should be $y = \frac{3}{2}(\sin(t) - \cos(t)) - 1 + Ce^{-t}$. How did we get this? *Recall that if we have a linear differential equation, the general solution can be written as the sum of the particular solution with the homogeneous part of the solution.*

To solve this directly, the integrating factor is e^t, so that:

$$\left(e^t y\right)' = 3e^t \sin(t) - e^t$$

To compute $\int e^t \sin(t)\,dt$, integrate by parts twice to get:

$$\int e^t \sin(t)\,dt = e^t \sin(t) - e^t \cos(t) - \int e^t \sin(t)\,dt$$

so that:

$$e^t y = \frac{3}{2}e^t\left(\sin(t) - \cos(t)\right) - e^t + C$$

so that $y = \frac{3}{2}(\sin(t) - \cos(t)) - 1 + Ce^{-t}$

12. Solve, and give the interval I on which the solution is valid: $\dfrac{dy}{dx} = \sqrt{1-y^2}$

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 1\,dx \Rightarrow \sin^{-1}(y) = x + C \Rightarrow y = \sin(x+C)$$

To be valid, $\sin^{-1}(y) = \theta \Rightarrow y = \sin(\theta)$ only for restricted values of θ (remember that the sine is only invertible if its domain is restricted). In this case, it is usual to take $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so the interval for x would be: $-\frac{\pi}{2} - C \le x \le \frac{\pi}{2} - C$

13. Solve:

(a) $\dfrac{dP}{dt} + 2tP = P + 4t - 2$ This is linear, so

$$P' + (2t-1)P = 2(2t-1) \Rightarrow e^{\int p(t)\,dt} = e^{t^2-t}$$

Therefore,

$$\left(e^{t^2-t}P\right)' = 2(2t-1)e^{t^2-t} \Rightarrow \text{ integrate with } u = t^2 - t \quad \Rightarrow \quad P = 2 + Ce^{-t^2+t}$$

(b) **The D.E. was changed to:** $(x+1)\dfrac{dy}{dx} + (x+2)y = 2xe^{-x}$

$$y' + \frac{x+2}{x+1}y = \frac{2x}{x+1}e^{-x}$$

Use long division or rewrite: $\frac{x+2}{x+1} = 1 + \frac{1}{x+1}$, and the integrating factor is:

$$e^{\int p(x)\,dx} = e^{x+\ln(x+1)} = (x+1)e^x$$

Therefore,

$$((x+1)e^x y)' = 2x \quad \Rightarrow \quad (x+1)e^x y = x^2 + C \quad \Rightarrow \quad y = \frac{x^2+C}{(x+1)e^x}$$

(c) $(y^2+1)\,dx - (y\sec^2(x))\,dy = 0$ This is a separable equation:

$$\frac{dy}{dx} = \frac{y^2+1}{y}\cdot\frac{1}{\sec^2(x)} \qquad \Rightarrow \qquad \int \frac{y}{y^2+1}\,dy = \int \cos^2(x)\,dx$$

To compute the integral on the left, use $u = y^2+1$ (u, du substitution), and for the integral on the right, use: $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ to get:

$$\frac{1}{2}\ln(y^2+1) = \frac{1}{2}(x + \frac{1}{2}\sin(2x)) + C \text{ or } \ln(y^2+1) = x + \frac{1}{2}\sin(2x) + C$$

(d) $(6x+1)y^2\frac{dy}{dx} + 3x^2 + 2y^3 = 0$ This is an exact equation:

$$(3x^2+2y^3)\,dx + (6x+1)y^2\,dy = 0$$

so $f_x = 3x^2 + 2y^3 \Rightarrow f(x,y) = x^3 + 2xy^3 + g(y)$. Now compare this f_y with $(6xy^2 + y^2)$,

$$6xy^2 + g'(y) = 6xy^2 + y^2 \Rightarrow g'(y) = y^2 \Rightarrow g(y) = \frac{1}{3}y^3$$

Putting it all together, the implicit solution is:

$$x^3 + 2xy^3 + \frac{1}{3}y^3 = C$$

(e) $xy^2y' = y^3 - x^3$ This is a homogeneous equation:

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{x}{y}\right)^2 = \frac{y}{x} - \left(\frac{1}{\frac{y}{x}}\right)^2$$

Let $u = y/x$, so that $\frac{dy}{dx} = \frac{du}{dx}x + u$,and:

$$\frac{du}{dx}x + u = u - u^{-2} \Rightarrow \frac{du}{dx}x = -u^{-2} \Rightarrow \int u^2\,du = -\int \frac{1}{x}\,dx$$

so that

$$\frac{1}{3}u^3 = -\ln|x| + C \Rightarrow \frac{1}{3}\frac{y^3}{x^3} = -\ln|x| + C \Rightarrow \qquad \frac{1}{3}y^3 = -x^3\ln|x| + Cx^3$$

(f) $y' + y = f(x)$, $y(0) = 1$, where $f(x) = \begin{cases} -1 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

This is actually two initial value problems:

$$y' + y = -1, \quad y(0) = 1 \quad \text{and} \quad y' + y = 1, \quad y(1) = ?$$

We will select $y(1)$ so that the overall function is continuous. The general solutions are, respectively,

$$y = -1 + C_1 e^{-x}, \qquad y = 1 + C_2 e^{-x}$$

The constant C_1 is found by using $y(0) = 1$, in which case $C_1 = 2$. Therefore, the solution when $0 \le x \le 1$ is given by:

$$y = -1 + 2e^{-x}$$

At time 1, we want our solutions to touch, so we will define $y(1) = -1 + 2e^{-1} = \frac{2}{e} - 1$. Now solve for C_2:

$$1 + C_2 e^{-1} = -1 + \frac{2}{e} \qquad \Rightarrow C_2 = 2 - 2e$$

The full solution is:

$$y(x) = \begin{cases} -1 + 2e^{-x} & \text{if } 0 \le x \le 1 \\ 1 + (2-2e)e^{-x}, & \text{if } x > 1 \end{cases}$$

14. If $y' + \dfrac{2}{x^2-1}y = 3x + 2$, $y(0) = y_0$, what is the largest interval on which the solution is defined? $-1 < x < 1$.