REVIEW QUESTIONS, EXAM 2, Math 244

1. True or False, and explain:

   (a) Let f and g be differentiable for every x. If the $W(f, g) = 0$ for every x, f, g must be linearly dependent.

   False. We did say that if $W(f, g) \neq 0$ at some x, the functions are linearly dependent on any interval containing that point. If the two functions are solutions to a linear homogeneous differential equation, and the Wronskian is zero on the interval where the solution is defined, then the statement would be true.

   (b) We cannot draw a direction field for a second order differential equation.

   True. The direction field depended on y' depending on t, y. In the second order case, y'' depends on t, y and y'.

   (c) Given that y_1 is part of the homogeneous solution, we can find both y_2 and y_p (at the same time) to $ay'' + by' + cy = f(x)$.

   True. We did this on some homework problems from Section 4.2.

   (d) We can always compute a fundamental set of solutions to $ay'' + by' + cy = 0$.

   True. Solve two initial value problems so that the Wronskian there is nonzero. A nice choice: Let y_1 solve the DE with initial conditions $y(0) = 1$, $y'(0) = 0$ and y_2 solve the DE with initial conditions $y(0) = 0$, $y'(0) = 1$ (therefore, the Wronskian at 0 is 1).

   (e) The Cauchy-Euler equation, $ax^2y'' + bxy' + cy = 0$ can be written as $a\hat{y}'' + b\hat{y}' + c\hat{y} = 0$ after an appropriate substitution (if True, write the substitution).

   True. Let $x = \mathrm{e}^t$, so $t = \ln(x)$. Furthermore, we make the substitutions:

   $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt} = \frac{1}{x}\dot{y}$$

   and

   $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d\left(\frac{dy}{dt}\right)}{dt}\frac{dt}{dx}\right) = \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right)$$

   (f) In using the Method of Undetermined Coefficients, is the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   This is false. We want to use: $y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$, which gives a different quadratic function for each of the sine and cosine function.

2. Suppose $W(f, g) = t^2 - 4$. What can we conclude about f, g?

   We can say that f, g are linearly independent functions. They could be linearly independent *solutions* on the interval $t < -2$, $-2 < t < 2$ or $t > 2$.

3. Without using the Wronskian, determine whether the given set of functions is linearly independent on the indicated interval:

   (a) $\ln(x)$, $\ln(x^2)$, $(0, \infty)$

   These are linearly dependent, since $\ln(x^2) = 2\ln(x)$, which is a constant multiple of the first function.

(b) x, $x+1$, $(-\infty, \infty)$

These are linearly independent, since they are not constant multiples of each other. In fact, the functions 1 and x are linearly independent as well.

(c) $xe^{x+1}, (4x-5)e^x, xe^x$, $(-\infty, \infty)$.

These are linearly dependent, since $xe^{x+1} = x\,(e^x e)$ is a constant multiple of the third function. (Note: If you have a list of functions, and two of them are linearly dependent, then the whole list is linearly dependent).

4. Finish the definition: Functions f, g are said to be Linearly Independent on an interval I if: *the only solution to* $c_1 f + c_2 g = 0$ *is the zero solution,* $c_1 = c_2 = 0$.

5. One way to obtain a fundamental set of solutions was to solve two initial value problems. How did we show that $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$ formed a fundamental set when $m = \alpha \pm \beta i$ was the solution to the characteristic equation?

   Given that $y_h = C_1 e^{(\alpha+\beta i)t} + C_2 e^{(\alpha-\beta i)t}$, we were given two sets of initial conditions so that the Wronskian was non-zero, and $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$. (You don't have to remember the initial conditions).

6. Suppose $m_1 = 3, m_2 = -5, m_3 = 1$ are the roots of multiplicity one, two and three respectively, of the characteristic equation. Write the general solution of the corresponding homogeneous linear DE if it is (a) an equation with constant coefficients, (b) a Cauchy-Euler equation.

   - For part (a), $y_h = C_1 e^{3t} + e^{-5t}\,(C_2 + C_3 t) + e^t\left(C_4 + C_5 t + C_6 t^2\right)$
   - For part (b), $y_h = C_1 x^3 + x^{-5}\,(C_2 + C_3\ln(x)) + x\left(C_4 + C_5\ln(x) + C_6(\ln(x))^2\right)$

7. Write down the general solution of the given differential equation using Method of Undetermined Coefficients (There will be two cases, $\omega = \alpha$ and $\omega \neq \alpha$). Do not solve for the coefficients: (a) $y'' + \omega^2 y = \sin(\alpha x)$, (b) $y'' - \omega^2 y = e^{\alpha x}$.

   - The roots to the characteristic equation are $m = \pm\omega i$.
     - If $\alpha \neq \omega$, $y = C_1\cos(\omega x) + C_2\sin(\omega x) + A\sin(\alpha x) + B\cos(\alpha x)$.
     - If $\alpha = \omega$, $y = C_1\sin(\omega x) + C_2\cos(\omega x) + x(A\sin(\omega x) + B\cos(\omega x))$.
   - The roots to the characteristic equation are $m = \pm\omega$.
     - If $\alpha \neq \omega$, $y = C_1 e^{\omega x} + C_2 e^{-\omega x} + Ae^{\alpha x}$.
     - If $\alpha = \omega$, $y = C_1 e^{\omega x} + C_2 e^{-\omega x} + Axe^{\omega x}$.

8. Find a linear second order differential equation with constant coefficients for which $y_1 = 1$ and $y_2 = e^{-x}$ are solutions to the homogeneous equation, and $y_p = \frac{1}{2}x^2 - x$ is a particular solution.

   From what is given, the characteristic equation is $m(m-1) = 0$, which corresponds to the homogeneous equation $y'' + y' = 0$. If we want $x^2 - x$ to be the particular solution, substitute it for y:

$$y_p'' + y_p' = 1 + (x-1) = x$$

   so that our final answer is: $y'' + y' = x$.

9. Write the solution in terms of α, then determine the value(s) of α so that $y(t) \to 0$ as $t \to \infty$:

$$y'' - y' - 6y = 0, \qquad y(0) = 1, y'(0) = \alpha$$

   The homogeneous solution is: $y_h = C_1 e^{3t} + C_2 e^{-2t}$. We get the following equations for the initial conditions, which we could solve by substitution, eliminating one variable, or by Cramer's Rule:

$$\begin{aligned} C_1 + C_2 &= 1 \\ 3C_1 - 2C_2 &= \alpha \end{aligned} \quad \Rightarrow C_1 = \frac{2+\alpha}{5}, \quad C_2 = \frac{3-\alpha}{5}$$

   For $y(t) \to 0$, $C_1 = 0$. Therefore, $\alpha = -2$.

10. Determine the longest interval for which the IVP is certain to have a unique solution. Do not attempt to find the solution:

$$t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, y'(3) = -1$$

First, put the D.E. in standard form:

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

which breaks the solution into one of the following intervals: (i) $t < 0$, (ii) $0 < t < 4$, or (iii) $t > 4$. Since the initial condition is set at $t = 3$, the longest interval is $0 < t < 4$.

11. Let $L(y) = y'' - 6y' + 5y$. Suppose that $y_{p_1} = 3\mathrm{e}^{2x}$ and $y_{p_2} = x^2 + 3x$ are particular solutions to $L(y) = -9\mathrm{e}^{2x}$ and $L(y) = 5x^2 + 3x - 16$. What is the particular solution to

$$y'' - 6y' + 5y = -10x^2 - 6x + 32 + \mathrm{e}^{2x}$$

Use the fact that this is a linear operator. Therefore, if

$$L(3\mathrm{e}^{3x}) = -9\mathrm{e}^{2x} \quad \Rightarrow \quad -\frac{1}{9}L(3\mathrm{e}^{3x}) = \mathrm{e}^{2x} \quad \Rightarrow \quad L\left(-\frac{1}{3}\mathrm{e}^{3x}\right) = \mathrm{e}^{2x}$$

and

$$L(x^2+3x) = 5x^2+3x-16 \quad \Rightarrow \quad -2L(x^2+3x) = -10x^2-6x+32 \quad \Rightarrow \quad L(-2x^2-6x) = -10x^2-6x+32$$

Therefore, $y_p = -\frac{1}{3}\mathrm{e}^{3x} - 2x^2 - 6x$.

12. Below you are given a differential equation and one of the homogeneous solutions. Use reduction of order to either find the other homogeneous solution, or both the homogeneous and particular solutions:

(a) $(1 - x^2)y'' + 2xy' = 0$, $y_1 = 1$
Using reduction of order,

$$\begin{array}{rl} y_2 = & u_1 y_1 \\ y_2' = & u_1' y_1 + u_1 y_1' \\ y_2'' = & u_1'' y_1 + 2u_1 y_1' + u_1 y_1'' \end{array} \quad \Rightarrow \quad y_1 = 1 \quad \Rightarrow \quad \begin{array}{rl} y_2 = & u_1 \\ y_2' = & u_1' \\ y_2'' = & u_1'' \end{array} \Rightarrow w' = \frac{-2x}{1-x^2}w$$

so that $w = 1 - x^2$, and $u_1 = x - \frac{1}{3}x^3$, which is also y_2, since $y_1 = 1$.

(b) $y'' - 3y' + 2y = 5\mathrm{e}^{3x}$, $y_1 = \mathrm{e}^x$ Using reduction of order,

$$\begin{array}{rl} 2y_2 = & 2u_1 y_1 \\ -3y_2' = & -3u_1' y_1 - 3u_1 y_1' \\ y_2'' = & u_1'' y_1 + 2u_1 y_1' + u_1 y_1'' \end{array} \quad \Rightarrow u_2'' y_1 + u_1'(2y_1 - 3y_1') = 5\mathrm{e}^{3x} \Rightarrow w' - w = 5\mathrm{e}^{2x}$$

Now,

$$\left(\mathrm{e}^{-x}w\right)' = 5\mathrm{e}^{x} \Rightarrow w = 5\mathrm{e}^{2x} + C\mathrm{e}^{x}$$

so that $u = \frac{5}{2}\mathrm{e}^{2x} + C\mathrm{e}^{x}$, and $y_2 = \frac{5}{2}\mathrm{e}^{3x} + C\mathrm{e}^{2x}$. Note that this combines the homogeneous part of the solution and the particular part of the solution.

(c) $4x^2y'' + y = 0$, $y_1 = x^{1/2}\ln(x)$
Note that $y_1' = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln(x) + 1\right)$. Setting $y_2 = u_1 y_1$, we get:

$$4x^2 y_2'' + y_2 = 4x^2 u_1'' y_1 + 8x^2 u_1' y_1' = 0$$

Substituting, and setting $w = u_1'$, we get:

$$w' = -\frac{4x^{3/2}(\ln(x)+2)}{4x^{5/2}\ln(x)}w \quad \Rightarrow \quad \frac{1}{w}\,dw = -\left(\frac{1}{x} + \frac{2}{x\ln(x)}\right)dx$$

(use integration by parts for that second integral) so that

$$\ln(w) = -\ln(x) - 2\ln(\ln(x)) \Rightarrow \ln(w) = \ln(x^{-1}) + \ln((\ln(x))^{-2}) \Rightarrow w = x^{-1}(\ln(x))^{-2}$$

Integrate by parts again to get $u = -\frac{1}{\ln(x)}$, so that $y_2 = -\frac{1}{\ln(x)} \cdot \sqrt{x}\ln(x) = \sqrt{x}$.

13. Referring to the previous problem, solve (a) by using a suitable substitution to make it first order, solve (b) by the Method of Undetermined Coefficents, and (c) by seeing it is a Cauchy-Euler equation.

    These will go a lot faster than Reduction of Order, but it was good for practice!

    - Let $w = y'$, so $w' = -\dfrac{2x}{1-x^2}w$ (which is what we had before). Now be sure and include the arbitrary constants so that we get all solutions:

    $$\ln(w) = \ln(1-x^2) + C_0 \Rightarrow w = C_1(1-x^2) \Rightarrow y' = C_1(1-x^2) \Rightarrow y = C_1(x - \frac{1}{3}x^3) + C_2$$

    - The homogeneous part is $y_h = C_1\mathrm{e}^{2x} + C_2\mathrm{e}^{x}$. We will guess that $y_p = A\mathrm{e}^{3x}$, giving us $A = \frac{5}{2}$.
    - Let $y = x^m$, and substitute into $4x^2y'' + y = 0$, giving us

    $$4m(m-1) + 1 = 0 \Rightarrow 4m^2 - 4m + 1 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$$

    (A repeated root). Therefore, $y_h = C_1x^{1/2} + C_2x^{1/2}\ln(x)$.

14. Solve. If given an IVP, solve for all unknown coefficients.

    (a) $y'' + 2y' + y = 0, y(0) = 1, y'(0) = 0$ $y(x) = \mathrm{e}^{-x}(1+x)$

    (b) $y'' - y = x + \sin(x)$, $y(0) = 2$, $y'(0) = -3$
    You should get that $y_h = C_1\mathrm{e}^{-x} + C_2\mathrm{e}^{x}$, and break the particular solutions up: $y_{p_1} = Ax + B$ and $y_{p_2} = A\sin(x) + B\cos(x)$. Solving, we get

    $$y(x) = \frac{1}{4}\mathrm{e}^{x} + \frac{7}{4}\mathrm{e}^{-x} - x - \frac{1}{2}\sin(x)$$

    (c) $y'' - y = \frac{2\mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^{-x}}$
    We might first simplify the right-hand-side (this would not be obvious from the start, but does make our simplifications easier later on...)

    $$\frac{\mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^{-x}} \cdot \frac{\mathrm{e}^x}{\mathrm{e}^x} = \frac{\mathrm{e}^{2x}}{1 + \mathrm{e}^{2x}}$$

    If $y_1 = \mathrm{e}^x, y_2 = \mathrm{e}^{-x}$, then the Wronskian is -2, and

    $$u_1' = \frac{-\mathrm{e}^{-x}\frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}}{-2} = \frac{\mathrm{e}^x}{1 + (\mathrm{e}^x)^2}$$

    To integrate, do a u, du substitution with $u = \mathrm{e}^x$ so that $u_1 = \tan^{-1}(\mathrm{e}^x)$ For the second function, we get (after some simplification):

    $$u_2' = -\frac{\mathrm{e}^{3x}}{1 + \mathrm{e}^{2x}}$$

Making the same substitution as before, $u = \mathrm{e}^x$,

$$u_2 = -\int \frac{u^2}{1+u^2}\,du = -\int 1 - \frac{1}{1+u^2}\,du$$

so that $u_2 = -\mathrm{e}^x + \tan^{-1}(\mathrm{e}^x)$. Put it all together for your final answer.

(d) $y'' - 2y' + y = x^2\mathrm{e}^x$

You should get $y_h = \mathrm{e}^x(C_1 + C_2x)$, and have an initial guess that $y_p = (Ax^2 + Bx + C)\mathrm{e}^x$. Comparing this to the homogeneous equation, we see that the final form for y_p is:

$$y_p = x^2(Ax^2 + Bx + C)\mathrm{e}^x = (Ax^4 + Bx^3 + Cx^2)\mathrm{e}^x$$

Things simplify quite a bit- We get $A = \frac{1}{12}$, and $B = C = 0$ giving the final answer as:

$$\mathrm{e}^x(C_1 + C_2x) + \frac{1}{12}x^4\mathrm{e}^x$$

(e) $x^2y'' - xy' + y = x^3$

The homogeneous part is $y_h = x(C_1 + C_2\ln(x))$. Put these into variation of parameters formula to get a particular solution of $\frac{1}{4}x^3$.

(f) $x^3y''' - 6y = 0$

In this case, the characteristic equation is $m^3 - 3m^2 + 2m - 6 = 0$. Note that this factors as: $m^2(m-3) + 2(m-3) = 0$ or $(m-3)(m^2+2) = 0$. Therefore, $m = 3, \pm\sqrt{2}\,i$, so our homogeneous solution is:

$$C_1x^3 + C_2\cos(\ln(x^{\sqrt{2}})) + C_3\sin(\ln(x^{\sqrt{2}}))$$

(g) $2x^2y'' + 5xy' + y = x^2 - x$

The homogeneous solution is $C_1x^{-1} + C_2x^{-1/2}$. You should find that the particular solution is $\frac{1}{30}x(2x-5)$.

(h) Solve by Variation of Parameters: $2y'' + y' - y = x + 1$

The homogeneous part of the solution is $C_1\mathrm{e}^{1/2x} + C_2\mathrm{e}^{-x}$. You should find that the particular part of the solution is $-x - 2$.

15. Be sure you can do the homework from Section 4.8!