Solving $ay'' + by' + cy = 0$

To solve $ay'' + by' + cy = 0$, we use the ansatz $y = \mathrm{e}^{\lambda t}$. Substitution of the ansatz into the ODE,

$$\mathrm{e}^{\lambda t}\left(a\lambda^2 + b\lambda + c\right) = 0$$

The equation $a\lambda^2 + b\lambda + c = 0$ is the characteristic equation associated with the DE. Its solutions are from the quadratic formula,

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A Summary of the Cases (details afterward):

1. If the roots are real, distinct: $y(t) = C_1\mathrm{e}^{\lambda_1 t} + C_2\mathrm{e}^{\lambda_2 t}$.
2. If we get a double root (in that case, $b^2 - 4ac = 0$, and $\lambda = -b/2a$, then we need a second linearly independent solution. We will show below that the general solution is:

$$y = C_1\mathrm{e}^{\lambda t} + C_2 t\mathrm{e}^{\lambda t} = \mathrm{e}^{\lambda t}\left(C_1 + tC_2\right)$$

3. If we get complex roots, $b^2 - 4ac < 0$, let $\lambda_{1,2} = \alpha \pm \beta i$, then:

$$y = \mathrm{e}^{\alpha t}\left(C_1\cos(\beta t) + C_2\sin(\beta t)\right)$$

Details: Case 2 In this case, $b^2 - 4ac = 0$, and we had only one real root, $\lambda = -b/2a$. We can obtain the second solution using variation of parameters (Section 4.2), and assume that the second solution is of the form $y_2 = u(t)\mathrm{e}^{\lambda t}$. Substituting y_2 into the ODE gives:

$$u''\left(a\mathrm{e}^{\lambda t}\right) + u'\left((b + 2a\lambda)\mathrm{e}^{\lambda t}\right) = 0$$

Divide by $a\mathrm{e}^{\lambda t}$. Substitute $\lambda = -b/2a$ to simplify:

$$u'' + u'\left(\frac{b}{a} + 2\lambda\right) = 0 \qquad \Rightarrow u'' + u'\left(\frac{b}{a} + 2\cdot\frac{-b}{2a}\right) = 0 \qquad \Rightarrow u'' = 0$$

Now take $u' = 1$ (we don't need a generic antiderivative- Choose one). And $u = t$. Therefore,

$$y_2(t) = u(t)\mathrm{e}^{\lambda t} = t\mathrm{e}^{\lambda t}$$

Details: Case 2 (Also see the Practice Sheet)
If the roots are complex, $b^2 - 4ac < 0$, we can write

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}\,i = \alpha \pm \beta i$$

From the Practice Sheet,

$$\mathrm{e}^{(\alpha \pm \beta i)t} = \mathrm{e}^{\alpha t}\mathrm{e}^{\pm \beta t\, i} = \mathrm{e}^{\alpha t}\left(\cos(\pm\beta t) + i\sin(\pm\beta t)\right) = \mathrm{e}^{\alpha t}\left(\cos(\beta t) \pm i\sin(\beta t)\right)$$

This last equality is because the cosine is an even function, $\cos(-w) = cos(w)$, and sine is an odd function, $\sin(-w) = -\sin(w)$.
If we let

$$y_1 = \mathrm{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right), \quad y_2 = \mathrm{e}^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$$

We know that we can find a fundamental set of solutions by taking the first function as the solution to:

$$ay'' + by' + cy = 0, \quad y(0) = 1, y'(0) = \alpha$$

and the second as the solution to:

$$ay'' + by' + cy = 0, \quad y(0) = 0, y'(0) = \beta$$

(Note that the Wronskian at $t = 0$ is $\beta \neq 0$, so that y_1, y_2 will form a fundamental set).
Let $y = C_1y_1 + C_2y_2$. In the practice problems, we showed that

$$y_1'(0) = \alpha + i\beta, \qquad y_2'(0) = \alpha - i\beta$$

For the case that $y(0) = 1$ and $y'(0) = 0$, we get the two equations:

$$C_1 + C_2 = 1, \qquad (\alpha + i\beta)C_1 + (\alpha - i\beta)C_2 = \alpha$$

and the solutions are: $C_1 = C_2 = \frac{1}{2}$, and $y = \mathrm{e}^{\alpha t}\cos(\beta t)$.
For the case that $y(0) = 0$ and $y'(0) = 1$, we get the two equations:

$$C_1 + C_2 = 0, \qquad (\alpha + i\beta)C_1 + (\alpha - i\beta)C_2 = \beta$$

and the solutions are: $C_1 = -\frac{i}{2}$, $C_2 = \frac{i}{2}$, and $C_1y_1 + c_2y_2$ simplifies to $\mathrm{e}^{\alpha t}\sin(\beta t)$. which is a multiple of just $\mathrm{e}^{\alpha t}\sin(\beta t)$.

Some Comments on Roots of Polynomials
The problem: $a_ny^n + \ldots + a_1y' + a_0y = 0$ will always reduce to solving the n^{th} degree polynomial:

$$a_n\lambda^n + a_{n-1}\lambda_{n-1} + \ldots + a_1\lambda + a_0 = 0$$

Notes:

- If $p_n(x)$ has a root at $x = a$, then $(x - a)$ is a factor. We could perform long division (or synthetic division) to factor this term out.
- If the coefficients of $p_n(x)$ are real, then the polynomial can be factored into a product of linear and irreducible quadratic terms. The roots of the quadratics will always come in complex conjugate pairs.
- If a real root a is repeated k times, the independent solutions will be:

  $$\mathrm{e}^{at}, t\mathrm{e}^{at}, \ldots, t^{k-1}\mathrm{e}^{kt}$$

  This will also work in the complex case- Multiply one fundamental set by t to get the second. For example, if the characteristic equation is: $(\lambda^2 + 1)^2 = 0$, a fundamental set would be:

  $$\sin(t), \cos(t), t\sin(t), t\cos(t)$$

- In general, it is difficult to find the roots of an n^{th} degree polynomial. There are formulas when $n = 2$, and $n = 3$. In Abstract Algebra, it is shown that there is no (closed form) formula if $n \geq 5$. In Maple, use the `roots` command.