Linear Systems Homework Solutions

Each list of 4 numbers are the 4 entries, a, b, c, d of a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ and classify the origin using our chart.

1. $-8, 18, -3, 7$

   SOLUTION: $\text{Tr}(A) = -1$, $\det(A) = -2$, $\Delta = 9$. From the chart, the origin is a SADDLE. The eigenvalues are $1, -2$, and the eigenvectors are $\langle 2, 1\rangle$ and $\langle 3, 1\rangle$. he solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^t + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathrm{e}^{-2t} = \begin{bmatrix} 2c_1\mathrm{e}^t + 3c_2\mathrm{e}^{-2t} \\ c_1\mathrm{e}^t + c_2\mathrm{e}^{-2t} \end{bmatrix}$$

2. $1, 1, -1, 1$

   SOLUTION: $\text{Tr}(A) = 2$, $\det(A) = 2$, $\Delta = -4$. From the chart, the origin is a SPIRAL SOURCE. The eigenvalue we'll work with is $1 + i$ and the eigenvector is from $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow v_2 = iv_1$$

   so an eigenvector is $\langle 1, -i\rangle$. We now compute $\mathbf{v}\mathrm{e}^{\lambda t}$ in preparation for the solution:

$$\mathbf{v}\mathrm{e}^{\lambda t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \mathrm{e}^t\mathrm{e}^{it} = \mathrm{e}^t \begin{bmatrix} \cos(t) + i\sin(t) \\ \sin(t) - i\cos(t) \end{bmatrix}$$

   The solution is:

$$\mathbf{x}(t) = c_1\text{Re}(\mathbf{v}\mathrm{e}^{\lambda t}) + c_2\text{Im}(\mathbf{v}\mathrm{e}^{\lambda t}) = \mathrm{e}^t \left(c_1 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \right)$$

3. $0, 1, -1, 2$

   SOLUTION: $\text{Tr}(A) = 2$, $\det(A) = 1$, $\Delta = 0$. From the chart, the origin is a DEGENERATE SOURCE with eigenvalue 1, eigenvector $\langle 1, 1\rangle$. The second eigenvector is found by: $(A - \lambda I)\mathbf{q} = \mathbf{v}$:

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow -q_1 + q_2 = 1 \Rightarrow \mathbf{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   The solution is:

$$\mathbf{x}(t) = c_1\mathbf{v}\mathrm{e}^{\lambda t} + c_2\mathrm{e}^{\lambda t}\left(t\mathbf{v} + \mathbf{q}\right) = \mathrm{e}^t \left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ t + 1 \end{bmatrix} \right)$$

4. $1, 1, -1, 3$

   SOLUTION: $\text{Tr}(A) = 4$, $\det(A) = 4$, $\Delta = 0$. From the chart, the origin is a DEGENERATE SOURCE with eigenvalue 2, eigenvector $\langle 1, 1 \rangle$. The second eigenvector is found by: $(A - \lambda I)\mathbf{q} = \mathbf{v}$:

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow -q_1 + q_2 = 1 \Rightarrow \mathbf{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   The solution is:

$$\mathbf{x}(t) = c_1 \mathbf{v} e^{\lambda t} + c_2 e^{\lambda t} (t\mathbf{v} + \mathbf{q}) = e^{2t} \left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ t+1 \end{bmatrix} \right)$$

   It's a coincidence that the eigenvectors in problems 3 and 4 are the same...

5. $-5, 1, -6, 0$

   SOLUTION: $\text{Tr}(A) = -5$, $\det(A) = 6$, $\Delta = 1$. From the chart, the origin is a SINK. The eigenvalues are $-2, -3$, and the eigenvectors are $\langle 1, 3 \rangle$ and $\langle 1, 2 \rangle$. The solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-3t} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-3t} \\ 2c_1 e^{-2t} + 3c_2 e^{-3t} \end{bmatrix}$$

6. $-1, 1, -5, 1$

   SOLUTION: $\text{Tr}(A) = 0$, $\det(A) = 4$, $\Delta = -16$. From the chart, the origin is a CENTER. The eigenvalue we'll work with is $2i$ and the eigenvector is:

$$\begin{bmatrix} -1-2i & 1 \\ -5 & 1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow v_2 = (1+2i)v_1$$

   so an eigenvector is $\langle 1, 1+2i \rangle$. We now compute $\mathbf{v}e^{\lambda t}$ in preparation for the solution:

$$\mathbf{v}e^{\lambda t} = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{2it} = \begin{bmatrix} \cos(2t) + i\sin(2t) \\ (\cos(2t) - 2\sin(2t)) + i(\sin(2t) + 2\cos(2t)) \end{bmatrix}$$

   The solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t)) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{bmatrix}$$

7. $4, -2, 1, 1$

   SOLUTION: $\text{Tr}(A) = 5$, $\det(A) = 6$, $\Delta = 1$. From the chart, the origin is a SOURCE. The eigenvalues are $2, 3$, and the eigenvectors are $\langle 1, 1 \rangle$ and $\langle 2, 1 \rangle$. The solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} c_1 e^{2t} + 2c_2 e^{3t} \\ c_1 e^{2t} + c_2 e^{3t} \end{bmatrix}$$

8. Let the matrix A be:

$$A = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}$$

where α is any real number. Let $\mathbf{x}' = A\mathbf{x}$, and classify the origin using the Poincare Diagram. For example, if $\alpha = 2$, we get a degenerate source.

We see that: $\text{Tr}(A) = \alpha$, $\det(A) = 1$ and $\Delta = \alpha^2 - 4$. In the Poincare Diagram, this means we are along a horizontal line at $\det(A) = 1$.

If $\alpha > 2$, $\Delta > 0$, and the origin is a SOURCE.

If $\alpha = 2$, $\Delta = 0$, and the origin is a DEGENERATE SOURCE.

If $0 < \alpha < 2$, $\Delta < 0$, and the origin is a SPIRAL SOURCE.

If $\alpha = 0$, $\text{Tr}(A) = 0$, and the origin is a CENTER.

If $-2 < \alpha < 0$, the origin is a SPIRAL SINK.

if $\alpha = -2$, the origin is a DEGENERATE SINK.

If $\alpha < -2$, the origin is a SINK.

9. Convert the following second order D.E.s to a system of first order, and solve using our current technique:

   (a) $y'' + 64y = 0$

   Let $x_1 = y$ and $x_2 = y'$. Then:

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -64y = -64x_1 \end{aligned} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -64 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   So $\text{Tr}(A) = 0$, $\det(A) = 64$, $\Delta = -256$. The origin is a CENTER, the eigenvalues are $\pm 8i$. Using $8i$, we get: $\langle 1, 8i \rangle$ for the eigenvector. Construct the solution:

$$\mathbf{v}e^{8ti} = \begin{bmatrix} 1 \\ 8i \end{bmatrix} (\cos(8t) + i\sin(8t)) = \begin{bmatrix} \cos(8t) + i\sin(8t) \\ -8\sin(8t) + i8\cos(8t) \end{bmatrix}$$

   The general solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos(8t) \\ -8\sin(8t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(8t) \\ 8\cos(8t) \end{bmatrix}$$

   In terms of the original problem, $y(t) = x_1(t) = c_1\cos(8t) + c_2\sin(8t)$.

   (b) $y'' + 5y' + 4y = 0$

   Let $x_1 = y$ and $x_2 = y'$. Then:

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -4y - 5y' = -4x_1 - 5x_2 \end{aligned} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $\mathrm{Tr}(A) = -5$, $\det(A) = 4$, $\Delta = 9$. The origin is a SINK, the eigenvalues are $-1, -4$ with eigenvectors $\langle -1, 1\rangle$, $\langle 1, -4\rangle$. The general solution: The general solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} \mathrm{e}^{-4t}$$

In terms of the original problem, $y(t) = x_1(t) = -c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-4t}$. Note that $x_2(t)$ is the derivative.

(c) $y'' - y' - 12y = 0$
Let $x_1 = y$ and $x_2 = y'$. Then:

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = 12y + y' = 12x_1 + x_2 \end{aligned} \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $\mathrm{Tr}(A) = 1$, $\det(A) = -12$, $\Delta = 49$. The origin is a SADDLE, the eigenvalues are $4, -3$ with eigenvectors $\langle 1, 4\rangle$, $\langle 1, -3\rangle$. The general solution:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \mathrm{e}^{4t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \mathrm{e}^{-3t}$$

In terms of the original problem, $y(t) = x_1(t) = c_1\mathrm{e}^{4t} + c_2\mathrm{e}^{-3t}$. Note that $x_2(t)$ is the derivative.