Linear Systems Overview and Homework

NOTATION: Capital letters will stand for matrices (like A), boldface letters are vectors (like $\mathbf{x}$), and small case letters, non-boldface, are scalars (like λ).
In this chapter (Chapter 8), we are focusing on solving the (linear) system of differential equations:

$$\mathbf{x}' = A\mathbf{x} \text{ or equivalently } \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\text{or equivalently } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We saw that, using the ansatz:

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

that $\lambda, \mathbf{v}$ must satisfy:

$$A\mathbf{v} = \lambda\mathbf{v}$$

This is the definition of the eigenvalues (λ) and eigenvectors ($\mathbf{v}$) of the matrix A.
This equation leads us to:

$$A\mathbf{v} - \lambda\mathbf{v} = 0 \quad \Rightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}$$

If $\det(A - \lambda I) \neq 0$, the only solution to this equation would be $\mathbf{v} = \mathbf{0}$. Therefore, to find λ, we must have:

$$\det(A - \lambda I) = 0$$

Which, for the 2×2 matrix A is:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

To find the eigenvalues of A, we must solve the quadratic:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \quad \Rightarrow \quad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}}{2}$$

We define the Discriminant, Δ as:

$$\Delta = \text{Tr}(A)^2 - 4\det(A)$$

If $\Delta > 0$, we get two real, distinct eigenvalues.
If $\Delta = 0$, we get one eigenvalue.
If $\Delta < 0$, we get complex conjugate eigenvalues.
As before, our solution will depend on Δ:

1. $\Delta > 0$ Find the corresponding eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$. The solution to $\mathbf{x}' = A\mathbf{x}$ is then:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 \mathrm{e}^{\lambda_1(t)} + c_2\mathbf{v}_2\mathrm{e}^{\lambda_2(t)}$$

2. $\Delta = 0$. We may or may not get two eigenvectors, but usually we will only get one. If we get two eigenvectors, see case (1). Otherwise, suppose we have only one and call it eigenvector $\mathbf{v}$.

   We suppose that the second solution is of the form: $\mathbf{x}_2(t) = (\mathbf{q} + t\mathbf{v})\mathrm{e}^{\lambda t}$. Substituting this into our differential equation yields:

$$\mathbf{x}_2'(t) = \mathrm{e}^{\lambda t}(\lambda\mathbf{q} + t\lambda\mathbf{v} + \mathbf{v}) \text{ and } A\mathbf{x}_2 = \mathrm{e}^{\lambda t}(A\mathbf{q} + tA\mathbf{v}) = \mathrm{e}^{\lambda t}(A\mathbf{q} + t\lambda\mathbf{v})$$

   Equating these and dividing by $\mathrm{e}^{\lambda t}$, and subtracting $t\lambda\mathbf{v}$ from both sides, we are left with:

$$A\mathbf{q} = \lambda\mathbf{q} + \mathbf{v} \qquad (A - \lambda I)\mathbf{q} = \mathbf{v}$$

   EXAMPLE:

$$A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$$

   The trace is -4, the determinant is 4, the discriminant is 0. The eigenvalue is $\lambda = -2$.

   The eigenvector $\mathbf{v}$ is found by solving $(A - \lambda I)\mathbf{v} = 0$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = v_2 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Now find $\mathbf{q}$: $(A - \lambda I)\mathbf{q} = \mathbf{v}$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad q_2 = 1 + q_1 \quad \Rightarrow \quad \mathbf{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   The overall solution:

$$\mathbf{x}(t) = c_1\mathrm{e}^{\lambda t}\mathbf{v} + c_2\mathrm{e}^{\lambda t}\left(t\mathbf{v} + \mathbf{q}\right) = c_1\mathrm{e}^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\mathrm{e}^{-2t}\begin{bmatrix} t \\ t+1 \end{bmatrix}$$

   (The origin in this case is a DEGENERATE SINK)

3. $\Delta < 0$. In this case, the eigenvector will also be complex. Just as in the second order case, we get the general solution as:

$$\mathbf{x}(t) = c_1\mathrm{Re}\left(\mathbf{v}\mathrm{e}^{\lambda t}\right) + c_2\mathrm{Im}\left(\mathbf{v}\mathrm{e}^{\lambda t}\right)$$

   EXAMPLE:

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

The trace is -2, the determinant is 5, the discriminant is -16. The eigenvalues are $-1 \pm 2i$. We only need one of these to find the eigenvector, so let $\lambda = -1 - 2i$.

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \Rightarrow 2iv_1 + 2v_2 = 0 \quad \mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Now expand $\mathbf{v}e^{\lambda t}$

$$e^{-t}(\cos(2t) - i\sin(2t)) \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{-t} \begin{bmatrix} \cos(2t) - i\sin(2t) \\ -\sin(2t) - i\cos(2t) \end{bmatrix}$$

So we get our two solutions:

$$\mathrm{Re}(e^{\lambda t}\mathbf{v}) = e^{-t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} \quad \text{and} \quad \mathrm{Im}(e^{\lambda t}\mathbf{v}) = e^{-t} \begin{bmatrix} -\sin(2t) \\ -\cos(2t) \end{bmatrix}$$

The general solution is now:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\sin(2t) \\ -\cos(2t) \end{bmatrix}$$

(The origin in this example is a SPIRAL SINK)

EXERCISES (TO REPLACE 8.2, 8.3)

Each list of 4 numbers are the 4 entries, a, b, c, d of a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ and classify the origin using our chart.

1. $-8, 18, -3, 7$
2. $1, 1, -1, 1$
3. $0, 1, -1, 2$
4. $1, 1, -1, 3$
5. $-5, 1, -6, 0$
6. $-1, 1, -5, 1$
7. $4, -2, 1, 1$

8. Let the matrix A be:

$$A = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}$$

where α is any real number. Let $\mathbf{x}' = A\mathbf{x}$, and classify the origin using the Poincare Diagram. For example, if $\alpha = 2$, we get a degenerate source.

9. Convert the following second order D.E.s to a system of first order, and solve using our current technique:

   (a) $y'' + 64y = 0$

   (b) $y'' + 5y' + 4y = 0$

   (c) $y'' - y' - 12y = 0$

   You might compare the solutions you get with the solutions we got earlier.