Method: "Brute Force Technique"

This method, about an ordinary point x_0, allows us to directly compute the coefficients of a power series solution up to any order. It relies on our ability to compute the n^{th} derivative at x_0 from the differential equation.

From the ansatz: $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, compare to the Taylor expansion of $y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$. This tells us that we can compute the nth coefficient of the power series by:

$$c_0 = y(x_0), c_1 = y'(x_0), c_2 = \frac{y''(x_0)}{2}, \ldots, c_n = \frac{y^{(n)}(x_0)}{n!}, \ldots$$

EXAMPLE: Compute the power series solution up to order 5 for the differential equation $y'' - 2xy' - 2y = 0$, $y(1) = 1$, $y'(1) = 1$.

Use the differential equation to compute the values of the derivatives:

$$y''(x) = 2xy'(x) + 2y(x) \Rightarrow y''(1) = 2(1) + 2(1) = 4$$

Differentiate to get a formula for $y'''(x)$:

$$y''' = 2y' + 2xy'' + 2y' = 4y' + 2xy'' \Rightarrow y'''(1) = 4(1) + 2(1)(4) = 12$$

Continue up to $y^{(5)}(1)$:

$$y^{(4)} = 4y'' + 2y'' + 2xy''' = 6y'' + 2xy''' \Rightarrow y^{(4)}(1) = 6(4) + 2(1)(12) = 48$$

$$y^{(5)} = 6y''' + 2y''' + 2xy^{(4)} = 8y''' + 2xy^{(4)} \Rightarrow y^{(5)}(1) = 8(12) + 2(1)(48) = 192$$

The solution up to 5th order is now:

$$y(x) = 1 + 1(x-1) + \frac{4}{2}(x-1)^2 + \frac{12}{3!}(x-1)^3 + \frac{48}{4!}(x-1)^4 + \frac{192}{5!}(x-1)^5 + h.o.t.$$

or simplified,

$$y(x) = 1 + (x-1) + 2(x-1)^2 + 2(x-1)^3 + 2(x-1)^4 + \frac{8}{5}(x-1)^5 + h.o.t.$$

In Maple, this can be done as:

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Order:=6;
de:=diff(y(x),x$2)-2*x*diff(y(x),x)-2*y(x)=0;
dsolve({de,y(1)=1,D(y)(1)=1},y(x),type=series);
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We can also analyze the relationship between the coefficients: First, note that:

$$y^{(n+1)}(1) = 2ny^{(n-1)} + 2y^n$$

Substitute: $y^{(n)} = n!c_n$:

$$(n+1)!c_{n+1} = 2n\,(n-1)!c_{n-1} + 2n!c_n \Rightarrow c_{n+1} = \frac{2}{n+1}\,(c_n + c_{n-1})$$

This is the recurrence relation for the coefficients.